WEB CHAPTER I

OPTIONS

...and Hedging

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This chapter provides a brief introduction to the most important aspects of the area of options. It covers options basics, arbitrage relationships, put-call parity, the Black-Scholes formula (and binomial option pricing), and less traditional applications of option pricing methods—but all in a very condensed form. You may prefer to resort to a full book on options and derivatives if this chapter is too telegraphic for you.

ANECDOTE: A Brief History of Options
Options have been known since Aristotle's time. The earliest known option contract is a real option. It was recorded by Aristotle in the story of Thales the Milesian, an ancient Greek philosopher. Believing that the upcoming olive harvest would be especially bountiful, Thales entered into agreements with the owners of all the olive oil presses in the region. In exchange for a small deposit months ahead of the harvest, Thales obtained the right to lease the presses at market prices during the harvest. As it turned out, Thales was correct about the harvest, demand for oil presses boomed, and he made a great deal of money.

A few thousand years later, in 1688, Joseph de la Vega describes in *Confusion de Confusiones* how options were widely traded on the Amsterdam Stock Exchange. It is likely that he actively exploited put-call-parity, an arbitrage relation between options discussed in this chapter. In the United States, options were traded over-the-counter since the 19th centry. A dedicated option market, however, was organized only in 1973.

Source: Wisegeek's "What Are Futures?"

1·1. Two Basic Derivatives: Call Options and Put Options

Base assets and Contingent Claims.

A **derivative** (also known as a **contingent claim**) is an investment whose value is itself determined by the value of some other underlying base asset. For example, a bet that a Van Gogh painting—the base asset—will sell for more than \$1 million is an example of a contingent claim, because the bet's payoffs are derived from the value of the Van Gogh painting (the underlying base asset). Similarly, a contract that states that you will make a cash payment to me that is equal to the square of the price per barrel of oil in 2010 is a contingent claim, because it depends on the price of the underlying base asset, oil. As with other financial contracts, we believe that both parties engage in derivatives contracts because it makes them better off ex-ante. For example, your car insurance is a contingent claim that depends on the value of your car (the base asset). Having the insurance is valuable only if the underlying base asset has been involved in an unfortunate accident. Ex-ante, both the insurance company and you are better off contracting to this contingent claim than you would be without the insurance contract. Ex-post, more than likely, only one of you will come out better off.

1·1.A. Basic Options You Already Know

Options are Contingent Claims.

An option is a specific type of contingent claim. Options give the owner the right, but not the obligation to do something (like buying shares at a given price) in the future. The value of this right will depend on the value of the base asset, which can fluctuate over time. In fact, options are so ubiquitous that, without using the designation, you have already worked with them in at least two contexts:

1. In Chapter **??**, on Page **??**, you computed the value of the levered equity ownership in a house. You can think of this levered equity ownership as the option to own the house *only* if it is worth enough to cover the creditors' claims. The levered equity owner possesses, *at his discretion*, the option of either walking away (best when the tornado hits and the house is worth less than the mortgage) or keeping possession of the house (best when the sun shines). The house itself is the base asset. In terms of net payoffs, the levered house owner receives the house value in excess of the mortgage only if the house turns out to be worth more than what the creditors are owed, and zero otherwise.

2. In Chapter **??**, on Page **??**, you learned that equity holders in corporations are also residual owners, but only *after* the corporate debt is paid off. Like a house owner, as a stockholder, you own the option to walk away from your investment. You do not have to make the creditors whole. Therefore, shareholders' equity is essentially an option on the value of the underlying base firm.

Anytime a contract includes limited liability, the limited liability holder has the option of walking away and abandoning—the financial contract is an option.

Call Options.

There is a common jargon when talking about options. A **call option** gives its holder the right to "call" (i.e., own) an underlying base security in exchange for a pre-specified payment, called the **strike price** or **exercise price**, usually for a specific period of time.

Example 1: Limited Liability you have already seen—equity and house.

Let's rephrase our house example from Chapter **??** in terms of option jargon. Recall that the house owner has the right to either pay the mortgage in period 1 or to just walk away. If the house value will be \$20,000, paying off the \$25,000 mortgage to own the \$20,000 would not be a good idea. If the house value will be above \$25,000, paying off the mortgage is a valuable option, exercise of which will give the homeowner net profits of the house value minus the mortgage value. If the house is worth \$100,000, exercise of the option means that the owner will receive a net of $\$100,000 - \$25,000 = \$75,000$. Thus, house ownership is a call option with a strike price of \$25,000. This can also be expressed as

$$\begin{aligned}\text{At Expiration: Homeowner's Value} &= \max(\,0\,,\ \text{House Value} - \$25,000\,)\\ \text{Homeowner's Value} &= \max(\,0\,,\ \text{House Value} - \text{Mortgage Value}\,)\end{aligned} \tag{I.1}$$

(The max function means "take whichever of its arguments is the bigger.") This formula is the payoff at expiration and intentionally does not take into account the upfront payment necessary to purchase the option (buy the house). Using the terminology of options does clarify the meaning of the **limited liability** that house equity owners enjoy: they can at most lose their upfront investments.

1·1.B. Stock Options

Example 2: Stock option.

Let's look at a common type of option that we have not yet discussed—stock call options. There are two basic types, calls and puts. If you purchase a July IBM stock call option with a strike price of \$85 on May 31, 2002, you get the right to purchase one share of IBM stock at the price of \$85 anywhere between then and July 2002. Call options increase in value as the underlying stocks appreciates and decrease in value as the underlying stock depreciates. If in June 2002, the price of a share of IBM stock were below \$85, your right will be worthless: shares will be cheaper to purchase on the open market. (Indeed, exercising would lose money: purchasing shares that are worth, say, \$70, for \$85 would not be a brilliant idea.) Again, the beauty of owning a call option is that you can just walk away. However, if in June 2002 the price of a share of IBM stock were to be above \$85, then your call option (purchase right) will be worth the difference between what IBM stock is trading for and your exercise price of \$85. You should exercise the right to purchase the share at \$85 from the call writer. For example, if the price of IBM stock were \$100, you would enjoy an immediate net payoff of \$100-\$85=\$15. The relationship between the call value and the stock value on the final moment at which the call option expires is

$$
\begin{aligned}
C_T(\,K = \$85,\ \text{at } T \text{ in June 2002} \Leftrightarrow \text{ remaining time } t \text{ of } 0\,) &= \max(\,0\,,\,S_T - \$85\,)\quad, \\
C_T(\,K, t = 0\,) &= \max(\,0\,,\,S_T - K\,)\quad,
\end{aligned}
\tag{I.2}
$$

where C is the value of the call option on the final date T, given the (pre-agreed) strike price K. So, if the stock price at expiration, S_T, is above K, the option owner earns the difference between S_T and K, and zero otherwise. Note that, like other derivatives, an option is like a side bet between two outside observers of the stock price. Neither party necessarily needs to own any stock. Therefore, because the person owning the call is paid $\max(\,0, S_T - K\,)$ at the final date (relative to not owning the call), the person having sold the call must pay $\max(\,0, S_T - K\,)$ (relative to not having written the call).

DIGGING DEEPER: *Stock ownership is itself an option on the underlying value of the assets. (The equity is valuable if the assets are worth more than the corporate debt.) An option on the stock of IBM is therefore really not just an option, but an option on an option!*

The price of the option compensates the option writer.

So, why would anyone sell ("write") an option? The answer is "for the money upfront." Table I.1 shows that the IBM call with a strike price of \$85 and expiration date of July 20 2002 would have cost you \$1.90 cents on May 31, 2002. (The IBM stock price then was \$80.50.) As long as the upfront price is fair—and many option markets tend to be very competitive and efficient—neither the purchaser nor the seller come out for the worse. Indeed, because both parties voluntarily engage in the contract, they should both come out better off ex-ante. Of course, ex-post, the financial contract will force one side to pay to the other, making one side financially worse off and the other side financially better off, relative to not having written the contract.

Put options

In some sense, a **put option** is the flip side of a call option. It gives you the right to "put" (i.e., sell) an underlying security for a specific period of time in exchange for a pre-specified price. For example, in May 2002, a particular put option may give you the right to sell a share of IBM stock at the price of \$75 up until July 20, 2002. Put options increase in value as the underlying stock depreciates. If in July the price of a share of IBM stock is above \$75, the right will be worthless: shares can be sold for more on the open market. However, if in June 2002 the price of a share of IBM stock is below \$75, the right will be worth the difference between \$75 and IBM's stock price. Here is a concrete example. If the IBM share price is \$50, as the put owner, you can purchase one share of IBM at \$50 on the open market and exercise your right to sell the share at $75 to the option writer for an immediate net payoff of $25. Unlike a call option, a put option speculates that the underlying security will decline in value. The relationship between the value of the put option and the stock value on the final moment of the expiration of the put can be written as

$$\begin{aligned} P_T(\,K = \$75,\ \text{at } T \text{ in July 20 2002} \Leftrightarrow \text{ remaining time } t \text{ of } 0\ ) &= \max(\,0\,,\ \$75 - S_T\,)\ , \\ P_T(\,K, t = 0\,) &= \max(\,0, K - S_T\,)\ . \end{aligned} \tag{I.3}$$

Common Uses.

Put options are often purchased as “insurance” by investors. For example, if you own a lot of IBM shares and IBM is trading at $80.50/share today, you may be willing to live with a little bit of loss, but not a lot. In this case, you may purchase put options that have a strike price of $75. If IBM were to end up at $60, the gain on your put option ($15/put) could make up for some of the losses ($20.50/share) of your underlying IBM shares. Of course, buying such put option insurance would cost you money today—$1.725 to be exact, according to Table I.1. In contrast, call options are often used to sell off “the upside.” For example, as the aforementioned large IBM share owner, you may decide that you like to keep the upside until $90, but that you do not care as much about the upside beyond $90 (or you believe that the price will not rise so high). In this case, you may sell a $90 call option today that gives you some immediate money ($0.725 to be exact) that you can invest anywhere (including into more IBM shares), but you would give up the extreme best payoffs above $90 until July 2002. The extra option fee of $0.725 would boost your return if the IBM stock price were to remain below $90. But if IBM were to end up at $120, you would only participate in the first $9.50 gain (from $80.50 to $90), of course plus the option fee with interest that you received upfront. The rest of the IBM upside would go to the call option purchaser. Why would someone want to purchase such a call option? It’s just another way to speculate that IBM will go up—and it is very efficient in terms of its use of cash money up front. The option to purchase IBM at $90 costs only $0.725 cents per share, much less than the $80.25 that IBM would cost. (Of course, the call is also a very risky proposition, which may easily end up with a −100% rate of return.)

American vs. European options

There are a variety of other option contract features. One common form, based on the time at which exercise can occur, is the **American option**. It allows the holder of the option to exercise the right any time up to and including the expiration date. A less common form is called a **European option**. (The S&P index options are European.) It allows the holder of the option to exercise the right only at the expiration date. The largest financial market for trading options on stocks is the **Chicago Board Options Exchange**, or **CBOE**. Its options are typically of the American type.

ANECDOTE: Geography and Options
The origin of the terms “European” and “American” is a historical coincidence, not a reflection of what kind of options are traded where. Although no one seems to remember the origins of these designations, one conjecture is that contracts called “primes” were traded in France. These could only be exercised at maturity—but they were not exactly what we now call European options. Instead, the option owner either exercised (and received $S - K$) or did not exercise and paid a “penalty” fee of D called a “dont” (not don’t). There was no up-front cost. (The best strategy for the prime owner was to exercise if $S - X > -D$.) Because these contracts could only be exercised at maturity and because American options could be exercised at any time, the terminology stuck.

Incidentally, “Bermuda options” or “Atlantic options” can be exercised periodically before maturity but not at any other time. They are so named not because they are used in Bermuda, but because Bermuda (and of course the Atlantic Ocean) lies between Europe and America.

Table I.1. Some Option Prices on May 31, 2002.

Underlying Base Asset		Expiration T	Strike Price K	Call Price	Put Price
IBM	\$80.50	Jul 20, 2002	\$85	\$1.900	\$6.200
		Different Strike Prices			
IBM	\$80.50	Jul 20, 2002	\$75	\$7.400	\$1.725
IBM	\$80.50	Jul 20, 2002	\$80	\$4.150	\$3.400
IBM	\$80.50	Jul 20, 2002	\$90	\$0.725	\$10.100
		Different Expiration Dates			
IBM	\$80.50	Oct 19, 2002	\$85	\$4.550	\$8.700
IBM	\$80.50	Jan 18, 2003	\$85	\$6.550	\$10.200

The source of these prices is OptionMetrics. July 20 would have been 34 trading days (out of 255/year) away. IBM's closing price at 4:30pm EST was 5 cents lower. The prevailing interest rates were 1.77% over 1 month, and 1.95% over 6 months. Note: Yahoo!*Finance* also reports option prices, e.g., for KO, look up http://finance.yahoo.com/q?s=ko&d=o

1·1.C. Splits and Dividends for Stock Options

Splits and Dividends?

What happens to the value of a common stock option when the underlying stock pays a **dividend** or executes a **stock split**? In a stock split, a company decides to change the meaning, but not the value, of its shares. For example, an owner who held 1,000 shares at \$80.50/share would in a 2-for-1 split own 2,000 shares. Splitting does not create shareholder value out of nothing—it should not change the market capitalization of the underlying company. Therefore, the resulting shares should be worth \$40.25/share post-split.

Splits are adjusted for, dividends are not.

Although such a split should make little difference to the owners of the shares (\$80,500 worth of shares, no matter what), it could be bad news for the owner of a call option. After all, a call with a strike price of \$75 would have been **in-the-money** (i.e., the underlying share price of \$80.50 was above the strike price) before the split. If the option were American, the call would be worth at least \$5.50 per share if exercised immediately. After the split, however, the call would be far **out-of-the-money** (i.e., the underlying share price of \$40.25 would be below the strike price of \$75). Fortunately, the option contracts that are traded on most exchanges (e.g., the CBOE) automatically adjust for stock splits, so that the value of the option does not change when a stock split occurs: in this case, the option's effective strike price would automatically halve from \$75 to \$37.50 and the number of calls would automatically double from 1 to 2. (Completing our option terminology, not surprisingly, **at-the-money** means that the share price and the strike price are about equal.)

But options are usually not adjusted for dividends.

But common options are typically *not* adjusted for dividend payments: if the \$80.50 IBM share were to pay out \$40 in dividends, unless money were to fall from heaven, the post-dividend share price value would have to drop to around \$40.50. Therefore, the in-the-money call option would become an out-of-the-money call. Consequently, if your call was American, you might decide to exercise your call with a \$75 strike price to net \$5.50 just before the dividend date.

Splits do not matter, dividends do matter.

In sum, when you purchase/value a typical stock option, you can ignore stock splits but not dividend payments of the underlying security.

An option contract is a bundle of 100 options.

Three final institutional details. First, because the value of options can be very small, e.g., 72.5 cents for each IBM call option, they are usually traded in bundles of 100. This is called an **option contract.** Five option contracts on IBM are therefore 500 options (options on 500 shares), which in the example would cost $\$0.725 \cdot 500 = \$362.50$. Second, CBOE options typically expire on the third Friday of each month, which is where the 20th of the month came from. Third, option prices that you read in the newspaper can be deceptive, because they are typically closing prices. However, the CBOE usually closes at 4:00pm CST/5:00 EST, which is half an hour later than the NYSE (4:30pm EST). This sometimes leads to seeming arbitrages in the newspaper, which are however non-existent because what really has happened is that the underlying stock price has changed. (In addition, newspapers may quote either recent bid or recent ask prices, rather than the price at which one can actually transact.)

1·1.D. Option Payoffs at Expiration

Payoff Diagrams

It is easiest to understand an option via its payoff table and payoff diagram. You have already seen these in the house and capital structure contexts. A payoff diagram shows the value of the option as a function of the underlying base asset on its final moment just before expiration. Figure I.1 shows the payoff tables and payoff diagrams for a call and a put option, each with a strike price of $90. The characteristic of any option's payoff is the kink at the strike price: for the call, the value is zero below the strike price, and a 45 degree line above the strike price. For the put, the value is zero above the strike price, and a 45 degree line below the strike price.

1·1.E. More Complex Option Strategies

Some common option strategies.

Multiple options are so often combined together that certain strategies have earned their own nicknames. If nothing else, discussing these positions gives you good exercises for graphing more complex payoff diagrams! The two basic kinds are spreads, which consist of long and short options of the same type (call or put), and combinations, which consist of options of different types.

Simple Spread A position that is long one option and short another option, on the same stock. The options are of the same type (puts or calls), have the same expiration date, but different strike prices. For example, a simple spread may purchase one put with a strike price of $90 and sell one put with a strike price of $70. Figure I.2 plots the payoff diagram for this position. You should confirm that it is correct by constructing your own payoff table.

Complex Spread (e.g., Butterfly Spread) Like a simple spread, but with more than one option. (You will get to graph the payoff diagram in one of the questions below.)

Straddle A straddle is the most popular combination. It combines one put and one call. (You will get to graph the payoff diagram in one of the questions below.)

A **calendar spread** is a position that is long one option and short another option, on the same stock. The options are of the same type (puts or calls), have the same strike prices, but different expiration dates. Therefore, they do not lend themselves to graphing in payoff diagrams, which hold the expiration date constant.

Solve Now!

Q I.1 *How is owning a call option the same as selling a put option? How is it different?*

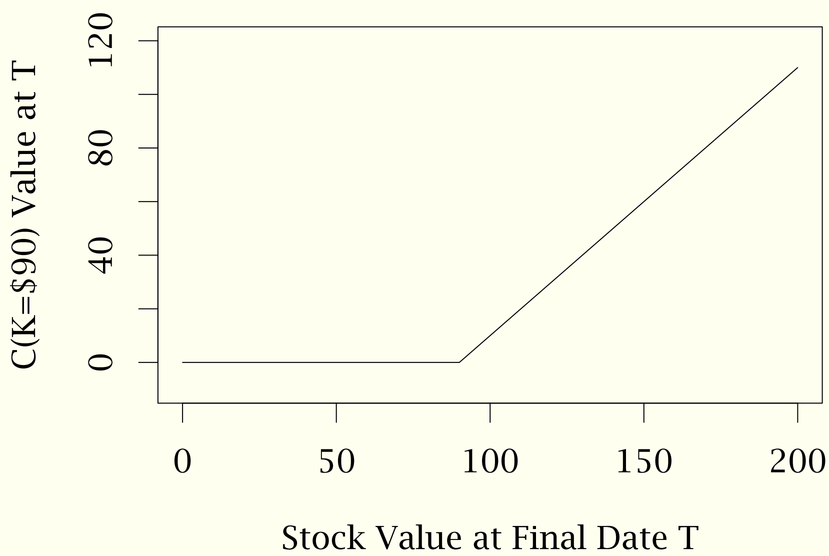
Q I.2 *Write down the payoff table and draw the payoff diagram (both at expiration) of a portfolio consisting of: one call option with a strike price K of \$60 and one put option with a strike price K of \$80.*

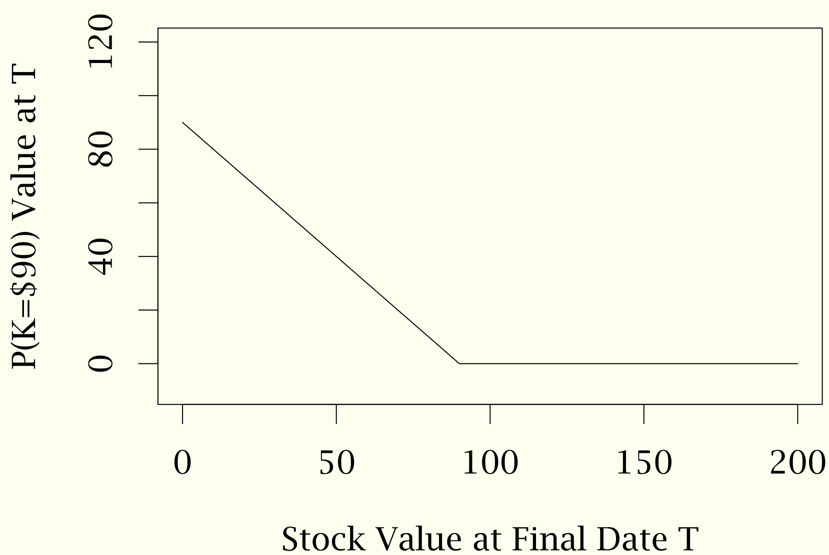
Q I.3 *Write down the payoff table and draw the payoff diagram (both at expiration) of a portfolio consisting of: one call short with a strike price K of \$60 and one put short with a Strike Price K of \$80.*

Q I.4 *Write down the payoff table and draw the payoff diagram (both at expiration) of a portfolio consisting of: one call short with a strike price K of \$60 and one put long with a Strike Price K of \$80.*

Figure I.1. Payoff Table and Payoff Diagram of Options with Strike Price $K = \$90$ on the Expiration Date T

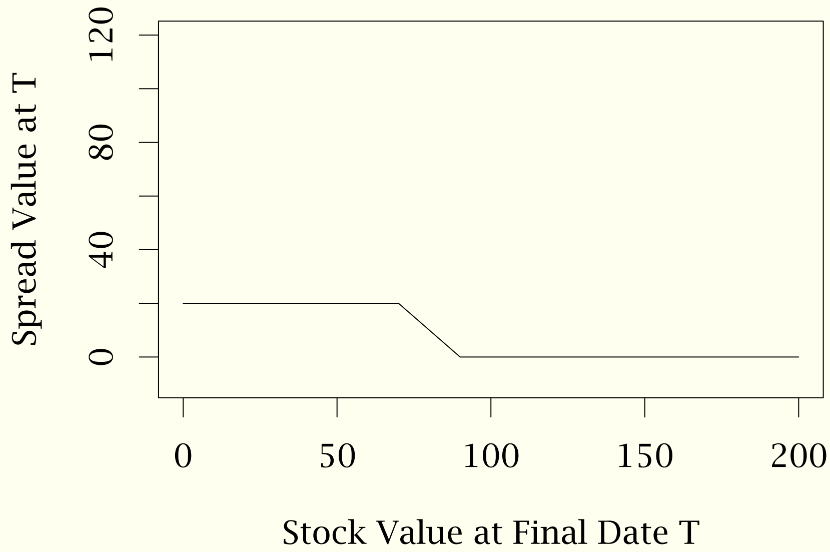
Stock_T	Call_T	Put_T	Stock_T	Call_T	Put_T
\$0	\$0	\$90	\$100	\$10	\$0
\$25	\$0	\$65	\$125	\$35	\$0
\$50	\$0	\$40	\$150	\$60	\$0
\$75	\$0	\$15	\$175	\$85	\$0





SIDE NOTE: Section 1·4 will graph the value of an option *prior to* expiration.

Figure I.2. Payoff Diagram of the Simple Spread



This spread is long one put option with a strike price of \$90 and short one put option with a strike price of \$70.

Q I.5 *Graph the payoff diagram for the following butterfly spread:*

- *One long call option with a strike price of \$50.*
- *Two short call options with strike prices of \$60.*
- *One long call option with a strike price of \$70.*

Q I.6 *Graph the payoff diagram for the following straddle:*

- *One long call option with a strike price of \$50.*
- *One long put option with a strike price of \$60.*

1·2. Static No-Arbitrage Relationships

We have already encountered arbitrage in other contexts (e.g., in Chapter ??). In the option pricing context, it is an especially powerful concept. To understand why arbitrage is so powerful in our option context, consider whether it is easier conceptually to value an underlying stock or an option on the underlying stock. For example, to value the underlying IBM stock, you have to determine all future cash flows of the underlying projects, and determine the appropriate costs of capital. You already know that this is very difficult. I cannot even easily tell you with great confidence that the price of an IBM share should be within any reasonable bounds, say between \$50 and \$150 (a factor three difference).

There are very few pricing bounds on underlying asset prices...

But with options, arbitrage can help us establish some strong bounds on the prices of options. One of the most important option pricing principles is that the price of an option should be such that no arbitrage is possible. You can construct clever portfolios—long in some securities, short in others—that are risk-free and therefore must not make money out of nothing. This will allow us to place bounds on appropriate prices of options today.

but there are many pricing bounds on their derivatives.

1·2.A. Some Simple No-Arbitrage Requirements

Let us derive our first bound: a call option cannot be worth more than the underlying base asset. For example, if IBM trades for \$80.50 per share, a call option with a strike price of, say, \$50 cannot cost \$85 per option. If it did, you should purchase the share, and sell the call. Today, you would make \$85 − \$80.50 = \$4.50. In the future, if the stock price goes up and the call buyer exercises, you deliver the one share you have, still having pocketed the \$4.50 net fee. If the stock price goes down and the call buyer does not exercise, you still own the share plus the upfront fee. Therefore, lack of arbitrage dictates that the value of the call C_0 today must be below the value of the stock S_0,

The value of the option depends on the stock, which makes it easier to determine.

$$C_0 \le S_0 \quad . \tag{I.4}$$

This is an upper bound on what a call can be worth, that firms up our knowledge of what a reasonable price for a call can be. It may be weak, but it at least exists—there is no comparable upper bound on the value of the underlying stock!

There are many other option pricing relations, which give us other bounds on what the option price can be today. Let us look at a few of them. For notation, $C_0(K,t)$ is the call option price today, K is the strike price, (lowercase) t is the time to option expiration, T is the time of expiration, and P_0 is the put option price today.

Selected Obvious No-Arbitrage Relations.

- Because the option owner only exercises it if it is not out-of-the-money, an option must have a positive value. Therefore,

$$C_0 \ge 0, \qquad P_0 \ge 0.$$

- It is better to have a call option with a lower exercise price. Therefore,

$$K_{\text{high}} > K_{\text{low}} \iff C_0(K_{\text{low}}) \ge C_0(K_{\text{high}}).$$

- It is better to have a put option with a higher exercise price. Therefore,

$$K_{\text{high}} > K_{\text{low}} \iff P_0(K_{\text{low}}) \le P_0(K_{\text{high}})$$

American options, which can immediately be exercised, enjoy further arbitrage bounds:

- The value of an *American* call today must be more than you can receive from purchasing it and exercising it immediately. Therefore,

$$C_0 \ge \max(0, S_0 - K)$$

- The value of an *American* put today must be more than you can receive from purchasing it and exercising it immediately.

$$P_0 \geq \max(0, K - S_0).$$

- It is better to have an *American* call option that expires later. Therefore,

$$t_{\text{longer}} > t_{\text{shorter}} \iff C_0(t = \text{longer}) \geq C_0(t = \text{shorter})$$

- It is better to have an *American* put option that expires later.

$$t_{\text{longer}} > t_{\text{shorter}} \iff P_0(t = \text{longer}) \geq P_0(t = \text{shorter})$$

These relationships are commonly called **no-arbitrage relationships**, for obvious reason.

1·2.B. Put-Call Parity

Put-call parity via Example.

There is one especially interesting and important no-arbitrage relationship, called **put-call parity**. It relates the price of a European call to the price of its equivalent European put, the underlying stock price, and the interest rate. Here is how it works. Assume that

- The interest rate is 10% per year;
- the current stock price S_0 is \$80;
- a 1-year European call option with a strike price of \$100 costs $C_0(K = \$100) = \$30$;
- and a 1-year European put option with a strike price of \$100 costs $P_0(K = \$100) = \$50$.

Further, assume that there are no dividends (which is important). Because the options are European, you only need to consider what you pay now, and what will happen at expiration T. (Nothing can happen in between.) If this was the situation, could you get rich? Try the position in Table I.2. (Note that any position that gives you a positive inflow today must give you a negative outflow tomorrow, or vice-versa. Otherwise, you would have something that always gave you or cost you money—not good if we want to believe in a sane world.)

Table I.2. Sample Put-Call Parity Violation

Today		At Final Expiration Time T					
		Covering S_T Range:	$S_T < \$100$		$S_T = \$100$	$S_T > \$100$	
Execute	Flow	Price S_T is:	\$80	\$90	\$100	\$110	\$120
Purchase 1 Call with strike price $K = \$100$	−\$30.00	You can exercise	\$0	\$0	\$0	+\$10	+\$20
Sell 1 Put with strike price $K = \$100$	+\$50.00	Your buyer can exercise	−\$20	−\$10	\$0	\$0	\$0
Sell 1 Share (= Short 1 Share):	+\$80.00	The short is unwound	−\$80	−\$90	−\$100	−\$110	−\$120
Save money, to pay strike price	−\$90.91	You get your money back	+\$100	+\$100	+\$100	+\$100	+\$100
Net=	+\$9.91	Net=	\$0	\$0	\$0	\$0	\$0

Not bad: You would earn \$9.91 today, and regardless of how the stock price turns out, you will not have to pay anything. This is an arbitrage. Naturally, we should not expect this to happen in the real world: one of the securities is obviously mispriced here. Given that the risk-free interest rate applies to all securities, and given the stock price is what it is, you can think of put-call parity as relating the price of the call option to the price of the put option, and vice-versa—and in this example, the call is just not expensive enough or the put is too expensive.

Table I.3. Algebraic Put-Call Parity Relation

Today		At Expiration	
	$T = 0$	$S_T < K$	$S_T > K$
Purchase 1 Call:	$-C_0(K)$	\$0	$S_T - K$
Sell 1 Put:	$+P_0(K)$	$S_T - K$	\$0
Sell 1 Share:	$+S_0$	$-S_T$	$-S_T$
Save to pay strike price:	$-\mathrm{PV}_0(K)$	$+\$K$	$+\$K$
Net	$-C_0(K) + P_0(K)$ $+S_0 - \mathrm{PV}_0(K)$	\$0	\$0

Put-call parity via Algebra.

Table I.3 repeats the example with algebraic variables instead of numerical values. The table shows that put-call parity means that it must be that

$$-C_0(K) + P_0(K) + S_0 - \mathrm{PV}_0(K) \ = \ 0 \ . \tag{I.5}$$

Let's apply this to the option prices in Table I.1. An IBM put with a strike price of \$85, expiring on July 20 2002, costs \$6.200. At an interest rate of 1.77% per annum, the present value of the strike price in 34 trading days would be \$84.80. Put-call parity states that the call should cost

$$\begin{aligned} C_0(K) &= P_0(K) - S_0 + \mathrm{PV}_0(K) \\ &= \$6.20 + \$80.50 - \$84.80 \\ &= \$1.90 \ . \end{aligned} \tag{I.6}$$

This was indeed the call price in the market, as you can see in Table I.1.

IMPORTANT: *Given an interest rate and the current stock price, the prices of a European call option and a European put option with arbitrary but identical expiration dates and strike prices, are related by* ***Put-Call Parity****,*

$$C_0(K) \ = \ P_0(K) + S_0 - \mathrm{PV}_0(K) \ . \tag{I.7}$$

The stock must not pay dividends before expiration.

1·2.C. Early Exercise and American Calls

Put-call parity, which holds only for options without dividends, implies that American calls are never exercised early.

Although put-call parity applies *only* to European options, it has an interesting and clever implication for American call options—they should never be exercised early. (Again, the underlying stock must not pay dividends.) The reason is that put-call parity indirectly establishes a relationships between the price of an American call option and a European call option. See, if an American call option is exercised immediately, it pays $C_0 = S_0 - K$. The question is, if the call is not exercised immediately, is the live option price more or less than this? Well, we know that the American option must be worth at least as much as the European option,

$$\text{American Call Value} \;\geq\; \text{European Call Value} \;= \tag{I.8}$$

and put-call parity tells us that

$$C_0 \;\geq\; P_0(\,K\,) + S_0 - \mathsf{PV}_0(\,K\,) \quad . \tag{I.9}$$

But, $P_0(\,K\,)$ is a positive number, and $\mathsf{PV}_0(\,K\,)$ is less than K, which means that

$$\begin{aligned} \text{American Call Value} \;\geq\; & \;\text{European Call Value} \\ = \;& P_0(\,K\,) + S_0 - \mathsf{PV}_0(\,K\,) \;\geq\; S_0 - \mathsf{PV}_0(\,K\,) \;\geq\; S_0 - K \quad . \end{aligned} \tag{I.10}$$

Therefore, the value of the American call option is always at least as much if it is not immediately exercised. You can usually make more money by selling the call in the market (at its arbitrage-determined value) than you can make by (stupidly) exercising it. Indeed, you can see for yourself in Table I.1 that the American option price is always higher than what you could get through immediate exercise. For example, the July 20, 2002 call with a strike price of \$75 could net you \$4.50 upon immediate exercise. However, you could sell the call for \$7.40 in the open market.

An American call is like a European call.

It follows that an American call option—even though it can be exercised before expiration—is not worth more than the equivalent European call option. That is, the value of the right to early exercise for an American call option is zero, and

$$\Longrightarrow \quad \text{American Call Value} \;=\; \text{European Call Value} \quad . \tag{I.11}$$

> **IMPORTANT**: *Assuming the underlying stock pays no dividends, put-call parity implies that the value of an American call option is higher live than if it is immediately exercised. Therefore, the American right to exercise early is worthless, and the price of a European call option is the same as the price of an American call option.*

DIGGING DEEPER: *The same cannot be said for the value of an American put option: It may be appropriate to exercise an American put early, which could pay up to $K - S_0$. Rearranging Formula I.7,*

$$P_0(\,K\,) \;=\; C_0(\,K\,) + [\,PV_0(\,K\,) - S_0\,] \quad . \tag{I.12}$$

Now, $P_0(\,K\,)$ can be either above or below $K - S_0$. Indeed, for far-in-the-money put options, e.g., after a large stock price decline, the call option is almost without value, and the put owner is better off exercising the put immediately and saving the resulting proceeds at the prevailing interest rate, instead of holding onto the put. In the real world, all put options that are far out-of-the-money have already been purchased and exercised before the final date, so they are no longer available.

That early exercise has no value can also not be said for call options if the underlying stock pays dividends. For example, if the underlying stock pays a liquidating dividend, and the call is in-the-money, it definitely becomes worthwhile for the American call option holder to exercise the call just before the dividend is paid.

Solve Now!

Q I.7 *List simple no-arbitrage relationships, preferably both in English and Algebra.*

Q I.8 *Write down the put-call parity formula, preferably without referring back. What are the inputs?*

Q I.9 *A 1-year call option with a strike price of $80 costs $20. A share costs $70. The interest rate is 10% per year. What should a 1-year put option with a strike price of $80 trade for?*

Q I.10 *(Continued.) How could you earn money if the put option traded in the market for $25 per share. Be explicit in what you would have to short and what you would have to long (buy).*

Q I.11 *Under what conditions can a European option worth as much as an American option?*

1·3. Valuing Stock Options Given Stock Prices

The B-S Formula—a dynamic arbitrage bound.

The preceding section was called "static arbitrage," because it relied on relations forced by no-arbitrage positions that would have to be established only once. We now look at the Black-Scholes Formula (honored with a **Nobel Prize**), which was developed in 1973 by Fischer Black and Myron Scholes. It computes the theoretical price of a European call option on a stock, utilizing the current stock price, the call option's strike price, the time to maturity, the risk-free interest rate, and the volatility (standard deviation) of a stock's rate of return. The idea behind the Black-Scholes Formula is a dynamic arbitrage that requires not just the establishment of a position once, but the constant rebalancing by an arbitrageur. The formula and the concepts behind it are among the most important advances of modern finance. However, the BS formula itself is only one among a number of formulas that can be built on the same concept.

Why you see a formula drop from the sky.

I am now breaking my rule—I do not first show you where the formula comes from, but just show you the formula and its use itself. This is because this is an introductory finance textbook, and you most likely will not have time to cover the underlying concepts until you take a full options course. (Still, the basic concepts are explained in the appendix.)

1·3.A. The Black-Scholes Formula

Let's get it over with!

The problem that the Black-Scholes formula solves for us is that, although we know the value of the call if we know the value of the put, or vice-versa, we need to pin down one in order to determine the other. So, we need a formula that values the European call option if all we have is the underlying stock price. This is what the Black-Scholes Formula gives us.

IMPORTANT: *The **Black-Scholes (B-S)** formula gives the value of a call option on a stock not paying dividends:*

$$\begin{aligned} C_0(\,S_0, K, t, r_{\mathcal{F}}, \sigma\,) &= S_0 \cdot \mathcal{N}(\,x_s\,) - PV_0(\,K\,) \cdot \mathcal{N}(\,x_k\,) \\ \text{where} \quad x_s &\equiv \frac{\log_e[S_0/PV_0(\,K\,)]}{\sigma \cdot \sqrt{t}} + {}^1/_2 \cdot \sigma \cdot \sqrt{t} \\ \text{and} \quad x_k &\equiv x_s - \sigma \cdot \sqrt{t} \end{aligned} \tag{I.13}$$

where

S_0	*is today's stock price,*
t	*is the time left to maturity,*
K	*is the exercise price,*
$PV_0(\,K\,)$	*is the present value of K that depends on $r_{\mathcal{F}}$ (the risk-free interest rate),*
σ	*is the standard deviation of the underlying stocks' continuously compounded rate of return, and it is often casually called just "the stock volatility." It is very similar to the stock's rate of return standard deviation, which we previously called Sdv.*
$\log_e(\,\cdot\,)$	*is the natural log, and*
$\mathcal{N}(\,\cdot\,)$	*is the cumulative normal distribution function (discussed below).*

The three parameters t, $r_{\mathcal{F}}$, and σ have to be quoted in the same time units. (Typically, they are quoted in annualized terms.)

DIGGING DEEPER: *The difference between Sdv (from Chapter (**bookc**:) **??** (**bookg**:) **??**) and σ is that the former is the standard deviation of the plain rate of return, while the latter is the standard deviation of the natural log of one plus the rate of return (the continuously compounded rates of return from Section **??**). For example, instead of computing the standard deviation over, say, two sample rates of return, +1% and −0.5%, σ would be computed from $\log_e(1 + 1\%) \approx 0.995\%$ and $\log_e(1 - 0.5\%) \approx -0.501\%$. These numbers are obviously similar for typical daily rates of returns that the difference can often be ignored. It is only for longer periods of time with larger rates of returns that the difference becomes important.*

An approximation formula allows us to obtain Cumulative Normal Distribution Function values.

The $\mathcal{N}(\,z\,)$ function is the **cumulative normal distribution** function. (Excel calls it "normsdist.") You can also look it up in Table **??** (Page **??**).

The B-S Formula works rather well in pricing real-world options.

Unlike the CAPM, which provides only modestly accurate appropriate expected rates of return, the Black-Scholes formula is usually pretty accurate in practice. The reason why it works so well is that it is built around an arbitrage argument, although not a static one. Trust me when I state that, as a potential arbitrageur, you can obtain the same exact rate of return as you would from the option if you purchased just the underlying stock and bonds in the right proportion and provided you could trade infinitely often. (This is explained in detail in Section 1·7.) In other words, if the call price does not equal the arbitrage stock plus bond price, you could get perfectly rich in a perfect market. In the imperfect real world, the two can diverge a little, but not beyond transaction costs. In contrast, if the CAPM Formula is not satisfied, you may find some great portfolio bets—but there are usually no arbitrage opportunities.

1·3.B. A Sample Application of the Black-Scholes Formula

Although the Black-Scholes formula may look awe-inspiring, it is not as painful as it appears at first sight. So, let us work our example that shows how you would use B-S to determine the price of the following 3-month call option:

The best way to understand how to use B-S is to just do it!

Stock Price Today	S_0	\$80.50
Agreed strike price	K	\$85.00
Time Remaining to Maturity, Per Annum	t	$^{17}/_{255}$
Interest Rate on Riskfree Bonds, Per Annum	$r_{\mathcal{F}}$	1.77%
Volatility (Standard Deviation) of underlying Stock, Per Annum	σ	30%

We want to determine the B-S call value:

$$C_0(\,S_0 = \$80.50, K = \$85, t = {}^{17}/_{255}, r = 1.77\%, \sigma = 30\%\,) \;=\; ? \;. \tag{I.14}$$

The necessary steps are:

1. Compute the present value of the strike price. For 34 out of 255 trading days, the interest rate would have been $(1+1.77\%)^{34/255} - 1 = 0.2342\%$. Therefore, the $\text{PV}_0(\,\$85\,) = \$84.80$.

2. Compute the input x_s, later needed as the argument in the left cumulative normal distribution function:

$$\begin{aligned} x_s &\equiv \frac{\log_e(S_0/\text{PV}_0(K))}{\sigma\cdot\sqrt{t}} + {}^1/_2\cdot\sigma\cdot\sqrt{t} \\ &= \frac{\log_e[\$80.50/\text{PV}_0(\$85)]}{30\%\cdot\sqrt{^{34}/_{255}}} + {}^1/_2\cdot 30\%\cdot\sqrt{^{34}/_{255}} \\ &\approx \frac{\log_e(\$80.50/\$84.80)}{30\%\cdot 0.365} + {}^1/_2\cdot 30\%\cdot 0.365 \\ &\approx \frac{\log_e(0.949)}{10.95\%} + {}^1/_2\cdot 10.95\% \\ &\approx \frac{-0.052}{10.95\%} + 5.48\% \\ &\approx -47.52\% + 5.477\% \\ &\approx -42.04\% \;. \end{aligned} \tag{I.15}$$

3. Compute x_k, the argument in the right cumulative normal distribution function:

$$\begin{aligned} x_k &\equiv x_s - \sigma\cdot\sqrt{t} \\ &\approx -42.04\% - 30\%\cdot\sqrt{^{34}/_{255}} \\ &\approx -42.04\% - 10.95\% \\ &\approx -53.00\% \;. \end{aligned} \tag{I.16}$$

4. Look up the standard normal distribution for these two arguments,

$$\mathcal{N}(\,-0.4204\,) \approx 0.3371 \quad , \quad \mathcal{N}(\,-0.5300\,) \approx 0.2981 \;. \tag{I.17}$$

5. Compute the B-S value,

$$\begin{aligned} & C_0(\,S_0 = \$80.50, K = \$85, t = {}^{34}/_{255}, r = 1.77\%, \sigma = 30\%\,) \\ = \;& S_0 \cdot \mathcal{N}(\,x\,) - \mathrm{PV}_0(\,K\,) \cdot \mathcal{N}(\,x - \sigma\cdot\sqrt{t}\,) \\ \approx \;& \$80.50 \cdot \mathcal{N}(\,-0.4204\,) - \$84.80 \cdot \mathcal{N}(\,-0.5300\,) \\ \approx \;& \$80.50 \cdot 0.3371 - \$84.80 \cdot 0.2981 \\ \approx \;& \$27.14 - \$25.28 \\ \approx \;& \$1.86 \;. \end{aligned} \tag{I.18}$$

In sum, a call option on a stock with a current price of \$80.50 with a strike price of \$85 and 34 trading days left to expiration should cost about \$1.86 if the underlying volatility is 30% per annum (and if the risk-free interest rate is 1.77% per annum). Now, trust me when I state that 30% was a reasonably good estimate of IBM's volatility in 2002. If you look at Table I.1, you will see that the actual call option price of just such an option was \$1.90, not far off from the theoretical B-S value of \$1.86.

DIGGING DEEPER: *If the stock pays dividends before expiration, you cannot use the B-S Formula. However, there are a number of ad-hoc adjustments that can "make it work." One method assumes that the call option would never be exercised, even just before a dividend date. (If the value of the dividend payments is relatively small compared to the value of retaining the option, this is a good assumption.) In this case, one can simply subtract the present value of all dividends remaining throughout the life of the option from the B-S stock price, before using it as an input into the B-S model. To learn more about dividend adjustments, you are best off to take a course in option pricing.*

1·3.C. The Black-Scholes Value for non-European Calls

B-S works for American calls.

You know that the B-S formula prices European call options for stocks without dividends. However, there are also American calls, European puts, and American puts. What about them?

The B-S formula works here.

American Calls You already learned in Formula I.11 that the value of an American call is equal to the value of a European call. Therefore, the B-S formula prices American call options just as well as European call options.

Given the call value, put-call parity pins down the put's value.

European Puts If you know the value of the European call option, you can use put-call parity (Formula I.7) to determine the value of a *European* put option with the same strike price and maturity as the call option as

$$\begin{aligned} P_0 &= \$1.86 - \$80.50 + \$84.80 \\ &= \$6.16 \\ P_0 &= C_0 - S_0 + \mathrm{PV}_0(\,K\,) \;. \end{aligned} \tag{I.19}$$

This happens to be close to but not exactly equal to the (American) put price of \$6.16 in Table I.1.

No easy solutions.

American Puts Unfortunately, there are no easy solutions to price American put options. You know that the American Put has to be worth at least as much as the European put, which you can price—but it could be worth more. You will see this for yourself in one of the exercises.

Solve Now!

Q I.12 *What is the value of a call option with infinite time to maturity and a strike price of zero? Use the parameters of our example:* $S_0 = \$80.50$, $r_{\mathcal{F}} = 1.77\%$, *and* $\sigma = 50\%$.

Q I.13 *(Excel) Write an Excel spreadsheet that computes the Black-Scholes Value as a function of its five inputs. This will teach you more about the Black-Scholes formula than all the pages in this book. Recall that the normal distribution function in Excel is* NORMSDIST.

Q I.14 *Price a call option with a stock price of \$80, a strike price of \$75, 3 months to maturity, a 5% risk-free rate of return, and a standard deviation of return of 20% on the underlying stock.*

Q I.15 *(Strike Price) Price our earlier call option but with a higher strike price. That is, price a call with a stock price of \$80, a strike price of \$80, 3 months to maturity, a 5% risk-free rate of return, and a standard deviation of return of 20% on the underlying stock.*

Q I.16 *(Interest Rate) Price our earlier call option with a higher interest rate. That is, price a call with a stock price of \$80, a strike price of \$75, 3 months to maturity, a 10% risk-free rate of return, and a standard deviation of return of 20% on the underlying stock.*

Q I.17 *(Volatility) Price our earlier call option with a higher volatility. That is, price a call with a stock price of \$80, a strike price of \$75, 3 months to maturity, a 5% risk-free rate of return, and a standard deviation of return of 30% on the underlying stock.*

Q I.18 *(Put) Price a put option with a stock price of \$80, a strike price of \$75, 3 months to maturity, a 5% risk-free rate of return, and a standard deviation of return of 20% on the underlying stock.*

Q I.19 *Price a straddle: one call and one put option on a stock with a price of \$80, both with strike prices of \$75, 3 months to maturity, a 5% risk-free rate of return, and a standard deviation of return of 20% on the underlying stock.*

Q I.20 *Price the same straddle, but with only 1 month left to maturity.*

Q I.21 *Price the same straddle, but at expiration.*

Q I.22 *Price a European IBM put option with a strike price of \$100, using the parameters of the example in the text:* $t = 34/255, r = 1.77\%, \sigma = 30\%$.

Q I.23 *(Continued) If the preceding American put option was priced at its Black-Scholes European equivalent, would holding onto it be optimal?*

1·4. The Black-Scholes Inputs

Let us look a bit closer at the five ingredients of the B-S formula.

1·4.A. Obtaining the Black-Scholes Formula Inputs

Only σ, the standard deviation of the rate of return on the underlying stock, is difficult to obtain.

The first four inputs, S_0, K, t, and $r_{\mathcal{F}}$, are either given by the option contract (the strike price K and time to expiration t); or can be easily found in the local newspaper (the current stock price S_0 and risk-free interest rate $r_{\mathcal{F}}$ [required to compute $\text{PV}_0(\,K\,)$]). Only one input, σ, the standard deviation of the underlying stock return, has to be guesstimated. There are two methods to do so.

1. The old fashioned way uses, say, three to five years of historical stock returns, and computes the standard deviation of daily rates of return:

$$\sigma_{\text{daily}} = \sqrt{\frac{\sum_{t=\text{Day }1}^{\text{Day N}} (r_t - \overline{r})^2}{N-1}} \quad . \tag{I.20}$$

   (To be perfectly accurate, the rates of returns that you should be using here are continuously compounded, not simple rates of return.) Then, this number is annualized by multiplying it by $\sqrt{252}$, 252 being the number of trading days. For example, if the daily standard deviation is 0.1%, the annual standard deviation would be $\sqrt{252} \cdot 0.1\% \approx 15.9\%$.

2. If other call option prices are already known, it is possible to extract a volatility estimate using the B-S Formula itself. For example, assume that the newspaper reports the price of the stock to be \$80.50 and the price of a July call with a strike price of \$80 to be \$4.150.

$$C_0(\,S_0 = \$80.50, K = \$80, t = 34/255, r = 1.77\%, \sigma = ?) \;=\; \$4.150 \quad . \tag{I.21}$$

   What is the volatility of the underlying stock consistent with the \$4 price? The idea is to try different values of σ until the Black-Scholes Formula exactly fits the known price of this option.

   Start with a volatility guess of 0.10. After tedious calculation, you find that

$$C_0(\,S_0 = \$80.50, K = \$80, t = 34/255, r = 1.77\%, \sigma = 0.20) = \$2.70 \quad . \tag{I.22}$$

   You know that option value increases with uncertainty, so this was too low a guess for σ. Try a higher value—say, 0.50:

$$C_0(\,S_0 = \$80.50, K = \$80, t = 34/255, r = 1.77\%, \sigma = 0.50) = \$6.18 \quad . \tag{I.23}$$

   Too high. Try something in between. (Because \$4.150 is closer than \$2.70 than to \$6.18, try something a little bit closer to 0.20, say 0.25.)

$$C_0(\,S_0 = \$80.50, K = \$80, t = 34/255, r = 1.77\%, \sigma = 0.25) = \$3.27 \quad . \tag{I.24}$$

   Still too high, but pretty close already. After a few more tries, you can determine that $\sigma = 0.325\cdots$ is the volatility that makes the B-S option pricing value equal to the actual call option price of \$4.15.

   You can now work with this **implied volatility** estimate as if it were the best estimate of volatility, and use it to price other options with the B-S formula. Unlike the historical estimated volatility, the implied volatility is forward-looking! That is, it is the market guess of what volatility will be like in the future.

   Obtaining an implied volatility is such a common procedure that many data bases will provide both the option price and an implied volatility. Indeed, so did OptionMetrics, which provided the option price data in Table I.1. Table I.4 adds OptionMetrics' reported

Table I.4. Adding Implied Volatilities to Table I.1

Underlying Base Asset		Expiration T	Strike Price K	Option Type	Option Price	Implied Volatility
IBM	$80.50	Jul 20, 2002	$85	call	$1.900	30.38%
IBM	$80.50	Jul 20, 2002	$85	put	$6.200	29.82%
		Different Strike Prices				
IBM	$80.50	Jul 20, 2002	$75	call	$7.400	34.89%
IBM	$80.50	Jul 20, 2002	$75	put	$1.725	34.51%
IBM	$80.50	Jul 20, 2002	$80	call	$4.150	32.58%
IBM	$80.50	Jul 20, 2002	$80	put	$3.400	31.67%
IBM	$80.50	Jul 20, 2002	$90	call	$0.725	29.24%
IBM	$80.50	Jul 20, 2002	$90	put	$10.100	29.18%
		Different Expiration Dates				
IBM	$80.50	Oct 19, 2002	$85	call	$4.550	31.32%
IBM	$80.50	Oct 19, 2002	$85	put	$8.700	31.61%
IBM	$80.50	Jan 18, 2003	$85	call	$6.550	31.71%
IBM	$80.50	Jan 18, 2003	$85	put	$10.200	31.40%

The source of both prices and implied volatilities is OptionMetrics. July 20 would have been 34 trading days (out of 255/year) away. IBM's closing price at 4:30pm EST was 5 cents lower. The prevailing interest rates were 1.77% over 1 month, and 1.95% over 6 months. Note: Yahoo!*Finance* also reports option prices, e.g., for KO, look up http://finance.yahoo.com/q?s=ko&d=o

implied volatilities. For our specific July $80 call, OptionMetrics computed an implied volatility of 32.58%—just about the 32.5% that you computed yourself.

Sometimes, this implied volatility is even used interchangeably with the option price itself. That is, instead of reporting the BS call price, traders might just say that the option is priced at a "32.5% vol." The advantage is that this allows traders to easily compare different options. Table I.4 shows that the $75 July call has a price of $7.40, while the $85 Jan put has a price of $10.20. How do you compare the two? Quoting them as volatilities makes them seem relatively more comparable.

The volatility "smile."

I noted earlier that the BS formula is not the only option pricing formula, though it is by far the most common, and perhaps the easiest to use. But there is some evidence that it may not be the correct formula. The prices of far out-of-the-money options, both calls and puts, tend to be too high. Put differently, according to the BS formula, out-of-the-money options are priced as if their volatility was higher than that of options that are at-the-money. If you draw the implied volatilities as a function of strike price, you therefore get a "volatility smile"—which is exactly what this empirical regularity is called by traders. One explanation for the smile is that there is a rare probability of a large shock that is ignored by the BS model, which is why far-out-of-the-money options are more expensive in the real world than in the model. This is especially plausible for puts, which can serve as insurance against a stock market crash, but less plausible for calls. This opens another question: there is no longer just one implied volatility for a stock, but different ones, depending on the strike price. Which of these should you use? To predict future volatility, the recommendation here is to use at-the-money options. They have tended to predict historical volatility better than out-of-the-money options.

I·4.B. Comparative B-S Statics

How the B-S Formula changes with its inputs.

If you have solved all the exercises from the previous section (as you should have before proceeding!), you have already seen how the Black-Scholes formula call option value changes with its inputs—and it better make sense to you. For example, you should realize that, everything else equal, a call option should be worth more if the stock price today is higher. Let's cover the five inputs systematically:

Stock Price Today (S_0): <u>Positive</u> A call option is worth more when the stock price today is higher. You actually saw this already as a static no-arbitrage relationship on Page 9, and the B-S formula obeys it! Furthermore, not only do we know that the B-S Formula increases with S, but we can even work out by how much. Look at the B-S Formula:

$$C_0(S, K, t, r, \sigma) = S \cdot \mathcal{N}(x_s) - \mathsf{PV}_0(K) \cdot \mathcal{N}(x_k) \; . \tag{I.25}$$

The stock price appears at this very high level, separate from the strike price K, and multiplied only by $\mathcal{N}(x_s)$. It turns out that $\mathcal{N}(x_s)$ is how the value of the call changes with respect to *small* value changes in the underlying stock price. For example, if $\mathcal{N}(x_s) = 0.6$, then for a ten cent increase in the value of the underlying stock, the value of the call option increases by six cents. Put differently, if you are long 100 shares and short 60 options, then your portfolio will not be affected one way or the other when the underlying stock price increases (or decreases) by 1 cent. You are said to be **hedged** against changes in the stock price; that is, your portfolio is insured against such changes. For this reason, $\mathcal{N}(x_s)$ is also called the **hedge ratio** (and sometimes just **delta**).

Given strike price (K): <u>Negative</u> A call option is worth more when the strike price is lower. Again, this was also a static arbitrage relationship.

Time Left To Maturity (t): <u>Positive</u> A call option is worth more when there is more time to maturity. Again, this was also a static arbitrage relationship.

Interest Rate to Maturity ($r_{\mathcal{F}}$): <u>Positive</u> A call option is worth more when the interest rate is higher. This comparative static is not as intuitive as the three previous "comparative statics." My best attempt at explaining this intuition is that as the call option purchaser, you do not need to lay out the cash to cover the strike price immediately. You almost live on "borrowed" time. The higher the interest rate, the more value there is to you, the call owner, not to have to pay the strike price upfront.

This is most obvious when the option is far in-the-money. For example, take a one-year option with a strike price of \$40 on a stock with a price of \$100. Assume that the volatility is zero. If the interest rate is 0, the value of the call option is \$60: with no volatility, you know that the option will pay off \$60, and with an interest rate of 0, the value of the future payoff is the same as its present value. However, if the interest rate is 20%, then you can invest the \$40 in bonds for 1 year. Therefore, the value of the option is \$60 (at exercise), plus the \$8 in interest earned along the way, a total of \$68.

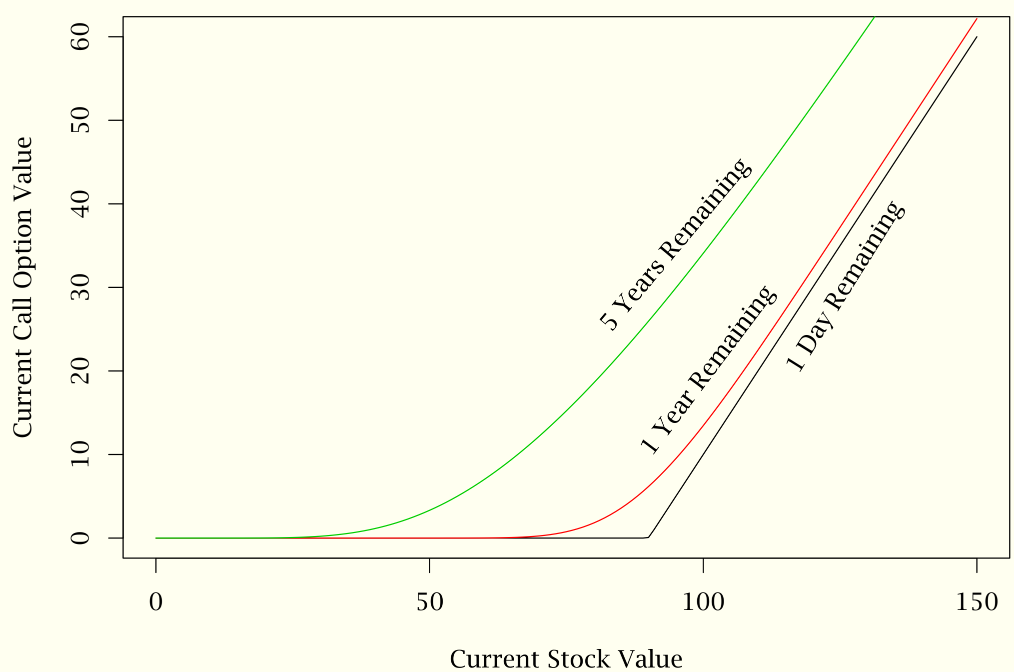
Volatility to Maturity (σ): <u>Positive</u> A call option is worth more when there is more volatility. When the underlying stock increases in volatility, the call option holder gets all the extra upside, but does not lose all the extra downside (due to limited liability). This increases the value of the option. If this comparative static is not obvious, then ask yourself whether you would rather own an option with a strike price of \$100 on a stock that will be worth either \$99 or \$101 at expiration, or on a stock that will be worth either \$50 or \$150 at expiration. Holding everything else constant, an option on a more volatile asset is worth more.

There is one counterintuitive feature of the B-S Formula: the expected rate of return on the underlying stock does not play a role in it. This is because the other inputs (the stock price, interest rate, and volatility) together already fully incorporate all the necessary information about the expected rate of return in the stock. Again, this is *not* intuitive. (Different purchasers can even disagree as to what the expected rate of return in the stock should be and still agree on the appropriate price on the option.) The mathematical reason why the expected rate of return on the stock does not enter the B-S formula is explained on Page 34.

But where is the expected rate of return on the stock in B-S?

1·4.C. Option Riskiness and Rates of Returns Prior to Expiration

Figure I.3. Black-Scholes Values Prior to and At Expiration



In this example, the time to expiration is either 0, 3 months, or 12 months. In all cases, the strike price is $K = \$90$, the annual interest rate is 5%, and the annual standard deviation is 20%.

The B-S Formula allows us to determine the price of a call option not only *on* the final expiration date (as we did on Page 6), but also *before* the final expiration date. Figure I.3 plots the B-S formula value of a call option with a strike price of $90, an interest rate of 5%, and a standard deviation of 20% for three different times to expiration. The Figure shows that the B-S value is always strictly above $\max(0, S_0 - K)$—otherwise, you could arbitrage by purchasing the call option and exercising it immediately. Moreover, you also already know that calls must be worth more when the underlying stock value is higher and when there is more time left to expiration. The figure nicely shows all these features.

The B-S value can be used to plot values prior to expiration.

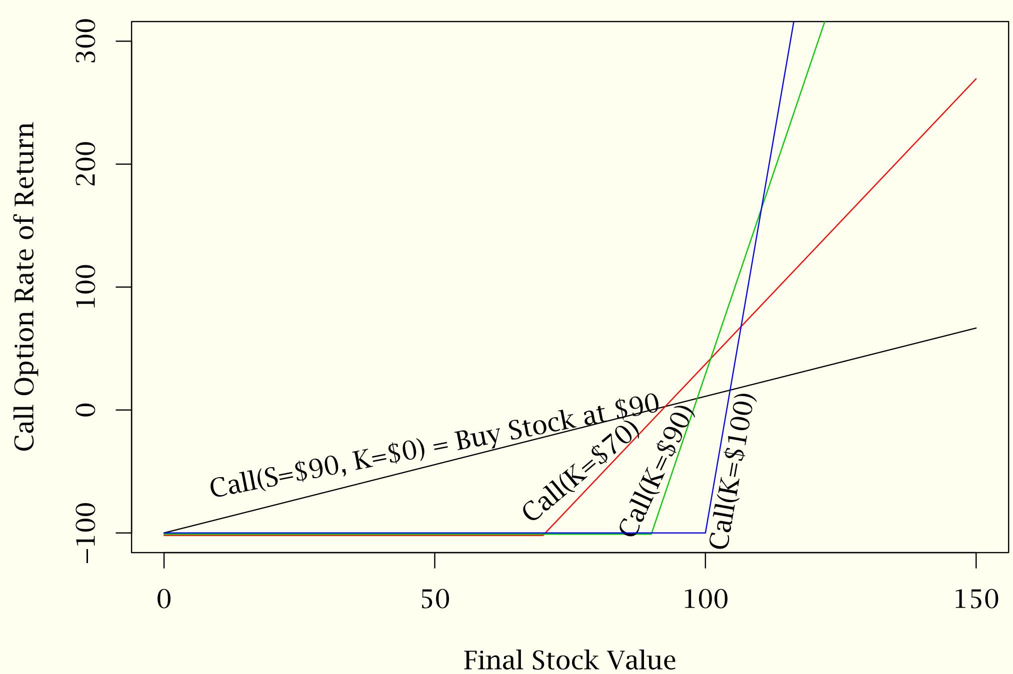
We can now ask another interesting question: What are the advantages and disadvantages of call options with different strike prices? The answer is that different options provide different risk profiles. For example, say the stock was trading at $100, 3 months prior to expiration; the annual interest rate was 5%; and the annual standard deviation of the stock's underlying rates of return was 20%. According to B-S, a call option with a strike price of $50 would have cost $50.61. A call option with a strike price of $90 would have cost $11.65. And a call option with a strike price of $120 would have cost $0.20. All are fair prices. But consider what happens if the stock were to end up either very, very high, or very, very low. If the stock price ends up

Options with different strike prices have different risk profiles.

at \$70, the \$50 option is the only one worth exercising, providing its holder with a \$20 payoff. This is equivalent to a rate of return of $(\$20 - \$50.61)/\$50.61 \approx -60\%$. Figure I.4 shows this calculation and a couple more. The call with the strike price of \$50 is relatively safe: it is in-the-money in both cases. The call with the strike price of \$90 has roughly a 50-50 chance of losing everything—but it provides more "juice" for each dollar invested *if* it expires in-the-money. Finally, the call with the strike price of \$120 is very likely to be a complete loss—but if the stock price were to exceed the strike price even by a little, the rate of return would quickly become astronomical. The rates of return on the three call options are graphed in Figure I.4.

Figure I.4. Rates of Returns on Call Option Investments

	Price Today	Stock will end at \$70		Stock will end at \$130	
	Payoff at T	Return	Payoff at T	Rate Return	Payoff at T
Call (Strike = \$0)	\$90.00	\$70-\$0	−22%	\$130-\$0	+44%
Call (Strike = \$50)	\$50.61	\$70-\$50	−60%	\$130-\$50	+58%
Call (Strike = \$70)	\$30.85	\$70-\$70	−100%	\$130-\$70	+94%
Call (Strike = \$90)	\$11.65	\$0	−100%	\$130-\$90	+243%
Call (Strike = \$100)	\$4.60	\$0	−100%	\$130-\$100	+552%
Call (Strike = \$120)	\$0.20	\$0	−100%	\$130-\$120	+4,942%
Call (Strike = \$130)	\$0.02	\$0	−100%	\$130-\$130	−100%



In all cases, the current stock price is \$100, the option is 3-months before expiration, the interest rate is 5%, and annual volatility is 20%.

Solve Now!

Q I.24 *In words, how does the value of a call option change with the Black-Scholes inputs?*

Q I.25 *Using the Excel spreadsheet you created earlier, graph the B-S value as a function of today's stock value for options with three different volatilities, 20%, 40%, and 80%. (Programming the formula just once yourself helps better in understanding it than reading about it a hundred times!) That is, repeat Figure I.4 for a 3-month option with strike price $K = \$90$, 3 months to expiration, and a 5% interest rate.*

Q I.26 *Using the Excel spreadsheet you created earlier, graph the B-S value as a function of today's stock value for options with three different interest rates, 5%, 10%, and 20%. That is, repeat Figure I.4 for a 1-year option with strike price $K = \$90$, 3 months to expiration, and a 20% volatility.*

Q I.27 *Advanced: There are numerous calculators on the World-Wide-Web that will calculate an implied volatility for you. It is also not too difficult to write one yourself in Excel, using the built-in equation solver. If you are familiar with Excel, create a spreadsheet that uses its equation solver to back out a volatility estimate, given a call price and the other B-S inputs.*

ANECDOTE: Executive Option Grants

Corporate boards often grant managers stock options with a strike price equal to the current underlying stock price in order to motivate them to work especially hard to increase the stock price. Aside, executive options have usually not been properly valued in corporate financial statements, which means that granting executive options had raised fewer eyebrows. Of course, even if not exercised, options can have tremendous value.

In April 2002, *Business Week* reported that Larry Ellison, CEO of Oracle, had pocketed $706 million from the exercise of long-held stock options, which exceeded the GDP of Grenada! "Fortunately," Oracle stock was off 57% that year, or Ellison's option value would have been $2 billion higher. The same year, Dennis Kozlowski, CEO of Tyco, hit number 3 on the executive payoff list. He fared even worse than Ellison: one year later, it appears he is headed for jail, partly for further criminally looting $600 million from Tyco. (Maybe the options were not high enough!) UPDATE THIS

1·5. Corporate Uses of Derivatives

Derivatives, such as options, are not just used to speculate, they can be used by corporations in many contexts, two of which we shall now discuss.

1·5.A. Corporate Hedging

Firms with global operations suffer exchange risk,

Sometimes, corporations or individuals want to avoid exposure to changes in the value of certain assets. For example, an American corporation may have sold some product to a German corporation for payment in Euros in six months. But the U.S. corporation may prefer to lock in the value of the Euro payment to be received in order to avoid the uncertainties of the exchange rate. After all, it needs to purchase its inputs in U.S. dollars today.

which they can hedge.

This can be done by **hedging** the exchange rate risk. The idea is to purchase a financial security that goes up by \$1 in value if the product payment in Euros goes down by \$1 in value (and vice-versa). For example, if there is a call option that increases in value by \$0.33 if the Euro increases in value by \$1, then the firm needs to sell three of these call options to neutralize its exposure. If the Euro goes up by \$1, then the underlying contract payments will go up by \$1 and the three call options will go down by $\$0.33 \cdot 3$. Conversely, if the Euro goes down by \$1, then the underlying contract payments will go down by \$1 and the three call options will go up by $\$0.33 \cdot 3$.

Hedging and option pricing rely on the same idea: how to mimic a payoff using another financial instrument.

Corporate hedging of uncertainties has become very common. The idea behind hedging is closely related to the idea of derivative securities: that is, a hedge ratio determines how many financial securities are required to neutralize the effect of changes in the value of an underlying asset.

An example where firm value correlates with a commodity price.

Here is another example of a hedge. Assume that you own a collagen factory. You have to purchase \$500,000 worth of collagen soon. Fortunately, from experience, you know that in the past, the publicly traded Acme skin-cream company correlated strongly with the price of collagen. Indeed, its public value was

$$\begin{aligned}&\text{Acme Skin Cream Value in million-\$} \approx \\ &10 - 0.5 \cdot (\text{Collagen Price in \$/pound}) + \sqrt{(\text{Collagen Price in \$/pound})} \quad .\end{aligned} \tag{I.26}$$

(Naturally, in the real world, you never have such a nice formula, but you could estimate it from the historical relationship between the value of Acme and the price of collagen.) The collagen price per pound is \$16/pound and Acme is worth $\$10 - 0.5 \cdot \$16 + \sqrt{\$16} = \$6$ million today. Right now, your collagen inputs cost \$8 million. How can you hedge your firm's collagen input price risk?

Determine how the value of Acme shares changes with the collagen price.

First, determine how the price of Acme changes with collagen:

$$\begin{aligned}\text{Acme } \$15 &\approx 10 - 0.5 \cdot \$15 + \sqrt{\$15} = \$6.37 \\ \text{Acme } \$16 &\approx 10 - 0.5 \cdot \$16 + \sqrt{\$16} = \$6.00 \\ \text{Acme } \$17 &\approx 10 - 0.5 \cdot \$17 + \sqrt{\$17} = \$5.62\end{aligned} \tag{I.27}$$

At \$12/pound, for each per-pound dollar change in the price of collagen, Acme's value changes by about \$375,000 in the opposite direction. Thus, if you own \$2.703 million of Acme shares and if collagen decreases in value by \$1/pound, then your stake increases in value by $\$2.703 \cdot 0.37 \approx \$1$ million. If you own \$1.35 million of Acme shares, your share stake changes in value by \$500,000 for each dollar change in collagen. Perfect: if you need to purchase \$500,000 of collagen soon and if you purchase \$1.35 million of Acme shares, then each dollar increase in the price of collagen increases both your Acme stock by \$500,000 and your collagen input cost by \$500,000. You would be hedged with respect to changes in the price of collagen.

Partial Hedging.

In the real world, you might hedge against some uncertainties, while leaving yourself exposed to other uncertainties. For example, you might purchase insurance against an earthquake, but not against a flood. Or firms may want to compensate their managers based only on their own firm's performance and not based on the industry's performance (which is beyond managers' control). One way to do this is to grant managers options or shares (long) in their own firm's stock price and write options or shares (short) in the industry's average stock price. Managers would do best if they outperformed their industry. Another way would be to grant them options for which the underlying base asset is the performance of the firm's stock minus the performance of the industry's stock. In both cases, managers would be hedged against industry risk and remain unhedged only with respect to their own firm's risk.

DIGGING DEEPER: *Recall from Page 20 that an option has a specific hedge ratio—and, indeed, the B-S Formula itself is derived from the idea that options must move together in a particular fashion with the underlying stock. (Section 1·7 makes this a bit clearer.)*

An easy and often functional way to obtain a hedge ratio is to estimate a regression, in which the dependent variable is the value V that is to be hedged, and the independent variable is the financial security H that is to be used for hedging:

$$V = \gamma_0 + \gamma_1 \cdot H + noise \quad . \tag{I.28}$$

The hedge ratio is the coefficient $\gamma_1 = {}^{\partial V}/_{\partial H}$. It measures how a (one-dollar) change in the underlying value translates into a (dollar) change in the financial instrument used for hedging.

Note that, as in the example, the hedge ratio could be different at different prices. If collagen had cost \$5/pound today, the change in Acme value with respect to a \$1 price per pound would be \$0.45 million, not \$0.37 million.

Why hedge?

You now know that firms can hedge to decrease underlying risks. (They can also trade derivatives or options to increase risks, however, and misleadingly call it hedging.) But should companies change their risks? In a Modigliani-Miller perfect capital market, hedging would not matter. Investors could themselves undo any hedge that the company could undertake—or take over the firm to eliminate any value-decreasing hedges that the firm would otherwise have undertaken. So hedging would be value-neutral. Indeed, it can even be shown that hedging is often equivalent to capital structure activity. The company could raise funds by selling new equity, sharing its upside, or by writing hedges, sharing its upside. It is only in an imperfect world that corporate hedging can matter. And now you have to start thinking about all the capital structure issues raised before—what tax implications does hedging have? does it reduce bankruptcy costs? does it induce the manager to make better or worse decisions? does it raise or hurt bondholder vs. stockholder conflicts?—and so on.

1·5.B. Synthetic Securities

How to make a put option yourself.

A different way to look at arbitrage relationships is to recognize that they define securities. That is, even if a put option were not available in the financial markets, it would be easy for you to manufacture one (assuming minimal transaction costs, of course). For example, return to the put-call parity relationship. It states that European options have the relationship

$$C_0(\,K\,) = P_0(\,K\,) + S_0 - \mathsf{PV}_0(\,K\,) \iff P_0(\,K\,) = C_0(\,K\,) - S_0 + \mathsf{PV}_0(\,K\,) \quad . \tag{I.29}$$

Instead of purchasing one put option, you can purchase one call option, short one stock, and invest the present value of the strike price in an account providing the risk-free rate of interest would pay. You would receive the same payoffs as if you had purchased the put option itself. Therefore, you have manufactured for yourself a synthetic put option.

ANECDOTE: Metallgesellschaft's Hedging
In Fall of 1993, **Metallgesellschaft** (a very large, hundred-year-old German company) experienced a major crisis: Owning a set of gas stations, Metallgesellschaft had agreed to purchase 2 billion barrels of oil at a price of \$16 to \$18 per barrel. Its presumed intent was to "hedge" its input costs. Unfortunately, it had not only mishedged (it did not have a need for 2 billion barrels in Fall 1993), but the oil price also moved against it: it had fallen to below \$15. Not surprisingly, the market value of all shares in Metallgesellschaft fell from about 3.7 billion DM to 1.5 billion DM.

Making and selling synthetic securities is big business on Wall Street.

Creating synthetic securities has become big business for Wall Street. For example, a company owning gas stations may wish to obtain an option to purchase 10,000 barrels of crude oil in 10 years at a price of \$50 per barrel. A Wall Street supplier of such call options models the price of oil, and determines the appropriate value of a synthetic call option. It then sells the call option to the firm for a little more. But would the Wall Street firm now not be exposed to changes in the oil price? Yes—but it would hedge this risk away. In our example, the Wall Street Firm would undertake a (usually dynamic) hedge. That is, it would first determine its hedge ratio, i.e., by how much the value of a synthetic 10-year call option with a strike price of \$50 per barrel changes with the underlying oil price today. This value may be 0.08. In this case, the Wall Street firm would purchase a contract for $10{,}000 \cdot 0.08 = 800$ barrels of oil. If the price of oil increases, then the Wall Street firm's own position in oil increases by the same amount as its obligation to the gas station company. This way, the Wall Street firm has no exposure to changes in the underlying oil price.

1·5.C. Strategic Options (Real Options)

There is also a complete chapter just on real options.

A real option depends not on an underlying financial asset (such as a stock), but on an underlying real asset.

Thinking of project option features in capital budgeting is essential. For example, firms have the ability to shut down production if the market price of their output product were to fall. That is, a project that allows management to curtail production when it is unprofitable is the equivalent of a call option with a strike price that depends on the price of the output good. Such options are called **real options**, because their exercise depends on the value of "real" assets, rather than on the value of financial assets. The house with a mortgage was a real option. Other real options, such as firms' options to increase or delay production, were described in Section **??** on Page **??**.

Sometimes, you can use the Black-Scholes formula to value the derivative.

If the base asset is a stock price in the future, we know the underlying stock price today and that the rate of return until expiration is roughly normally distributed. If the underlying base asset value (e.g., the house or the firm's output product or the firm's input product [e.g., gold or oil]) is similarly roughly normally distributed, we can even use the Black-Scholes formula to value the derivative asset. Doing so replaces the need to estimate the appropriate cost of capital, $\mathcal{E}(\tilde{r})$, on the project with the need to estimate the variance, $\sigma(\tilde{r})$. (Usually, estimating volatility is easier and more reliable.)

The method of valuation is less important than the recognition of the real option.

However, the most important aspect of real options is to recognize their presence, not the method of valuation. Whether the B-S Formula is used to avoid estimating the appropriate cost of capital, or a CAPM type formula with an expected cash flow estimate is used to obtain the appropriate cost of capital, is of relatively less importance. The big mistakes that managers often commit is that they fail to value the real option *at all*.

Solve Now!

Q I.28 *Assume that oil is trading for \$20 per barrel today. Your refinery is working on producing fuel as we write this. It will be worth \$1,500,000 if oil will trade at \$20/barrel, \$1,800,000 if oil will trade at \$25/barrel, and \$2,100,000 if oil will trade at \$30/barrel. How would you hedge, and what is the value of your fuel?*

Q I.29 *You have received an offer to buy a lease for one week's worth of production (100 ounces) in a particular gold mine. This lease will occur in exactly 18 months. It is an old mine, so it*

ANECDOTE: Environmental Options

Publicly traded options exist not only on stocks. For example, there is an active market in pollution options, which give option owners the legal right to spew out emissions such as CO_2. Experts generally agree that despite some shortcomings, the system of permitting trading in pollution rights and derivatives thereon has led to a cleaner environment. It is no longer in the interest of a polluter to maximize pollution: shutting down an old plant and selling the right to pollute can be more profitable than operating the plant.

costs \$400/ounce to extract gold from this mine. Gold is trading for \$365/ounce today, but has a volatility of 40% per annum. The prevailing interest rate is 10% per year. What is the value of the gold mine?

Q I.30 *Now assume that you own this mine. If the mine is inexhaustible, but can only extract 100 ounces per week, and the production cost increases by 20% per year (starting at \$400 next week, your first production period), how would you value this mine? (Do not solve this algebraically. Just think about the concepts.)*

1·6. Summary

The chapter covered the following major points:

- The contract features of options: call options give the right to purchase underlying securities at a predetermined strike price for a given period of time. Put options give the right to sell underlying securities at a predetermined strike price for a given period of time. American call options give this right all the way up to the final expiration; European call options give this right only at the final expiration.
- Option payoffs at expiration, and complex option strategies are best graphed with payoff diagrams.
- A number of static no-arbitrage relationships limit the range of prices that an option can have.
- The most important no-arbitrage relationship is put-call parity, which relates the price of a call to the price of a put, the price of the underlying stock and the interest rate.
- Put-call parity implies that American call options are never exercised early, and therefore that American calls are worth the same as European options. (This assumes no dividends.)
- The Black-Scholes Formula relates the price of a call to five input parameters. The B-S value increases with the stock price, decreases with the strike price, increases with the time left to maturity, increases with the volatility, and increases with the risk-free interest rate.
- Corporations can use derivative-related concepts to hedge their risk; Wall Street can use derivative-related concepts to create synthetic securities.

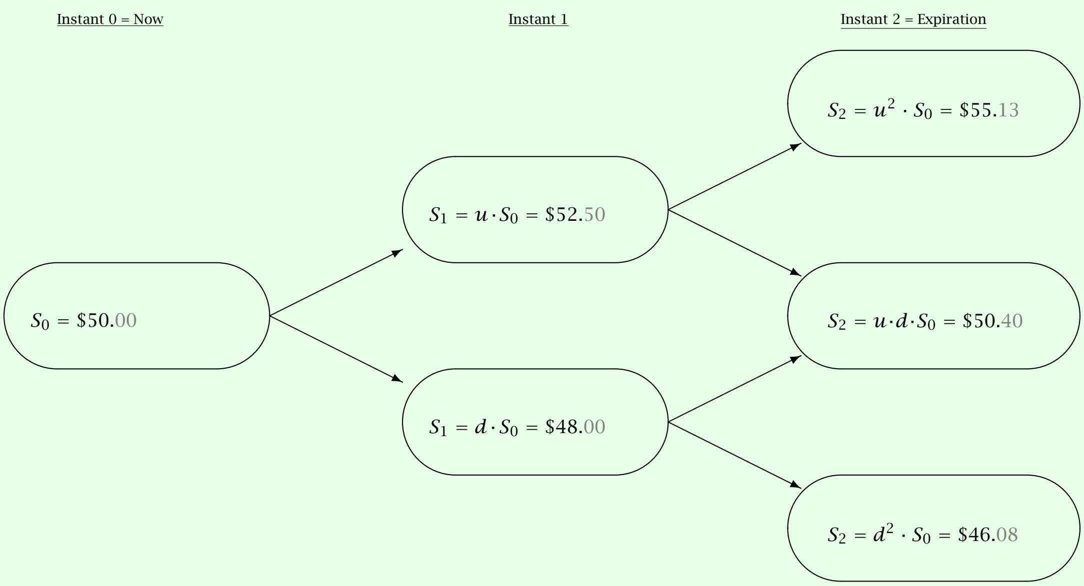
I·7. Advanced Appendix: The Ideas Behind the Black-Scholes Formula

In the previous sections, you learned how to use the Black-Scholes formula. However, it descended on you out of the ether. If you are wondering where the formula actually comes from, then this section is for you.

I·7.A. Modelling the Stock Price Process as a Binomial Tree

Each instant, stock prices goes up by u or down by d

To price an option, we first need to determine how to model the stock price process. The basic building element for the Black-Scholes Formula is the assumption that over one instant, the stock price can only move up or down. So, over two instants, the stock price can move up twice, move up once and move down once, or move down twice. Use the letter u to describe the stock price multiplier when an up move occurs, and d to describe the stock price multiplier when a down move occurs. You can represent the stock price process with a binomial tree. For example, if $d = 0.96$ (which means that on a down move, the stock price declines by 4%) and $u = 1.05$ (the stock price increases by 5%), the stock price is as follows:



There are more outcomes in the middle of a binomial tree than at the edges.

The middle outcome occurs on two possible paths, while the two extreme outcomes occur only on one path each; $u \cdot d \cdot S_0$ can come about if there is one u followed by one d, and if there is one d followed by one u. In real life, the stock price is really a tree with many more than just 2 levels, so we will want to generalize this process. For example, if there are 10 instants, what would be the worst possible outcome? Ten instant down movements mean that the stock price would be

$$\text{Worst Case Scenario:} \quad d^{10} \cdot S_0 \;=\; 0.96^{10} \cdot \$50 \;=\; \$33.24 \;. \tag{I.30}$$

The second-worst outcome would be 1 instant of up-movement, and 9 instants of down-movement.

$$\text{Second-Worst Case Scenario:} \quad d^{9} \cdot u^{1} \cdot S_0 \;=\; 0.96^{9} \cdot 1.05^{1} \cdot \$50 \;=\; \$36.36 \;. \tag{I.31}$$

Although the worst scenario can only occur if there are exactly ten down-movements, there are ten different ways to fall into the second-worst scenario, ranging from $duuuuuuuuu$,

$uduuuuuuuu$,...,to $uuuuuuuuud$. This should bring back bad memories of your S.A.T. test: these are the ten possible combinations, better written as

$$\begin{aligned} \binom{10}{1} &= \frac{10!}{1! \cdot 10!} = 10 \\ \binom{N}{i} &= \frac{N!}{N! \cdot (N-i)!} \end{aligned} \tag{I.32}$$

Therefore, with N levels in the tree, the stock price will be $u^i \cdot d^{N-i} S_0$ in $\binom{N}{i}$ paths. The probability of exactly 1 in 10 up moves, if the probability of each up move is 40%, would be

$$\begin{aligned} \mathcal{P}rob(\text{1 u's, 9 d's}) &= \binom{10}{1} \cdot 0.4^1 \cdot (1-0.4)^{10-1} \approx 4\% \\ \mathcal{P}rob(\text{(i) u's, (N-i) d's}) &= \binom{N}{i} \cdot p^i \cdot (1-p)^{N-i} \quad . \end{aligned} \tag{I.33}$$

1·7.B. Matching a Stock Price Distribution To A Binomial Tree

With many nodes, a binomial tree becomes a log-normal distribution.

Why do we work with this unrealistic binomial tree process, given that a future stock price is more likely to have a bell-shaped distribution? Put differently, how realistic is this binomial process? Figure I.5 plots the distribution of prices at the end of the tree if there are 1,000 nodes, if up and downs are equally likely, and if $u = 0.999$ and $d = 1.001$. This binomial process looks as if it can generate a pretty reasonable distribution of possible future stock price outcomes.

DIGGING DEEPER: *If we assume that the stock prices can only move up or down each instant and that there are an infinite number of instants, then the underlying stock price follows a **Log-Normal Distribution**, with −100% as the lowest possible outcome.*

How to fit parameters.

A practical question is how to select u, d, and q (q is the true probability of an up-move, not the "pseudo-probability" of an up-move discussed later) in a simulated tree to match an empirically observed stock price distribution. Assume you have a historical rate of return series to provide you with a reasonable mean and a reasonable variance for the expected rate of return. Call "dt" a really tiny time interval. Now, select u and d as follows:

$$u = m \cdot dt + s \cdot \sqrt{dt} \quad , \qquad d = m \cdot dt + s \cdot \sqrt{dt} \quad . \tag{I.34}$$

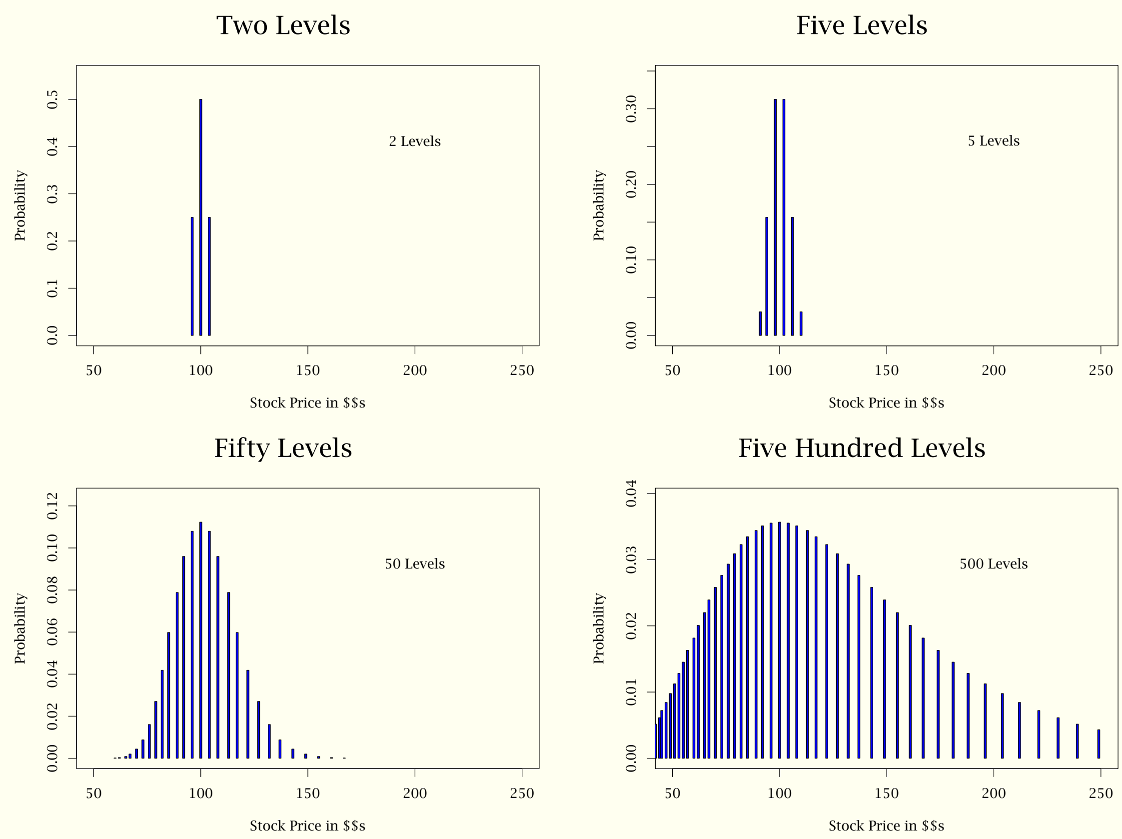
In the limit, these choices create a log-normal distribution, which is completely characterized by its mean and variance, with mean m and standard deviation s.

1·7.C. The Option Hedge

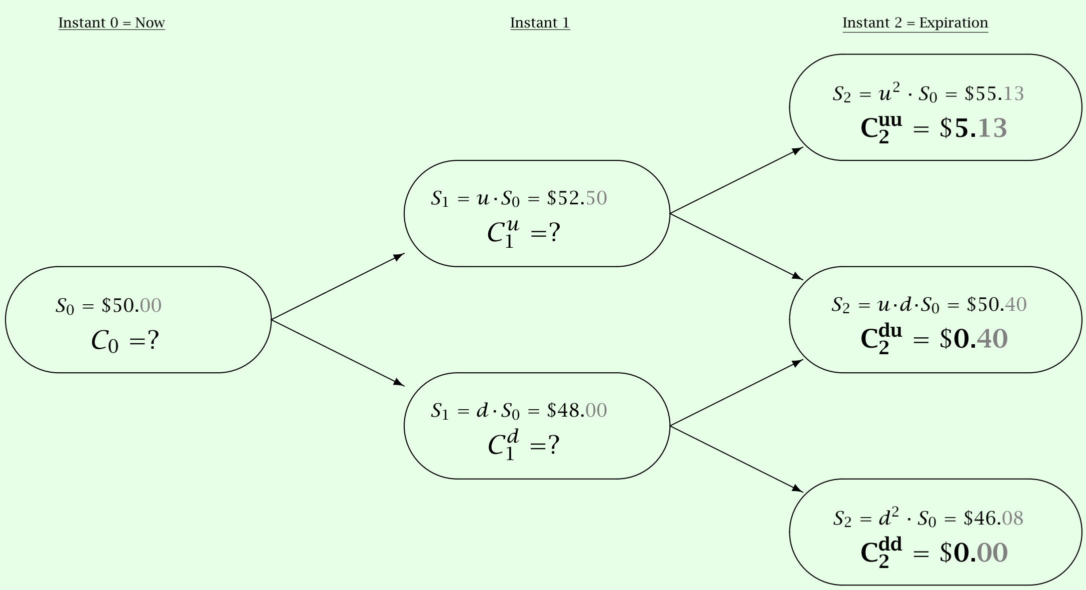
First, set out our goal: the value of an option two instants before expiration.

Assume that the stock price follows a binomial tree process, and that d and u are known. Your goal is to determine the value of an option. Start small: Price an option with a strike price of \$50 over these last two instants before expiration. On inspection of the tree, the option pays \$0 if the stock price moves down twice, \$0.40 if the stock price moves up once and down once (or the opposite), and \$5.13 if the stock price moves up twice.

Figure I.5. Stock Price Processes Simulated Via Binomial Processes



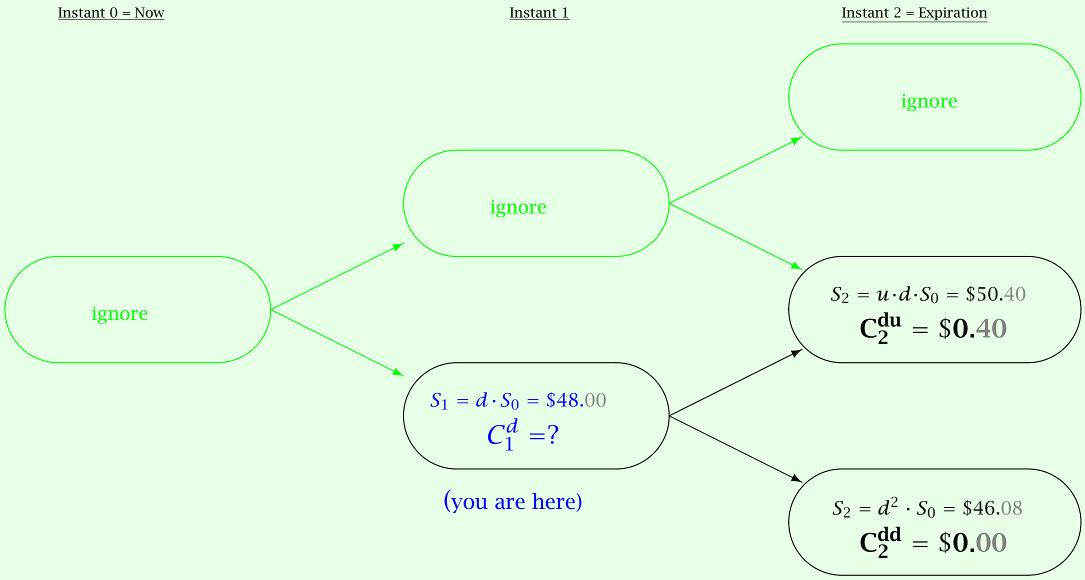
The probability of an up move at each tree node is 50-50. The value multiplier is $u = 1.01$ if an up-move occurs, $d = 1/1.01$ if a down move occurs. The figures differ in the number of levels in the tree, in clockwise order: 2, 5, 50, and 500.



Your ultimate goal is to determine the call price at the outset, C_0.

First place yourself into the position where the stock price has moved down once already, i.e., where the stock price stands at \$48.00.

Price the lower triangle first.



Your immediate goal is to buy stocks and risk-free bonds so that you receive \$0 if the stock moves down and \$0.40 if the stock moves up. Assume you purchase δ stocks and b bonds. Bonds increase at a risk-free rate of 1.001 (i.e., $1 + r$) each instant. If you own δ stock and the stock price goes up, you will own $\delta \cdot u \cdot S_0$ stock. If you own δ stock and the stock price goes down, you will own $\delta \cdot d \cdot S_0$ stock. Can you purchase a particular δ amount of stock and a particular b amount of bonds to earn exactly the same as your call option? Solve for b and δ so that

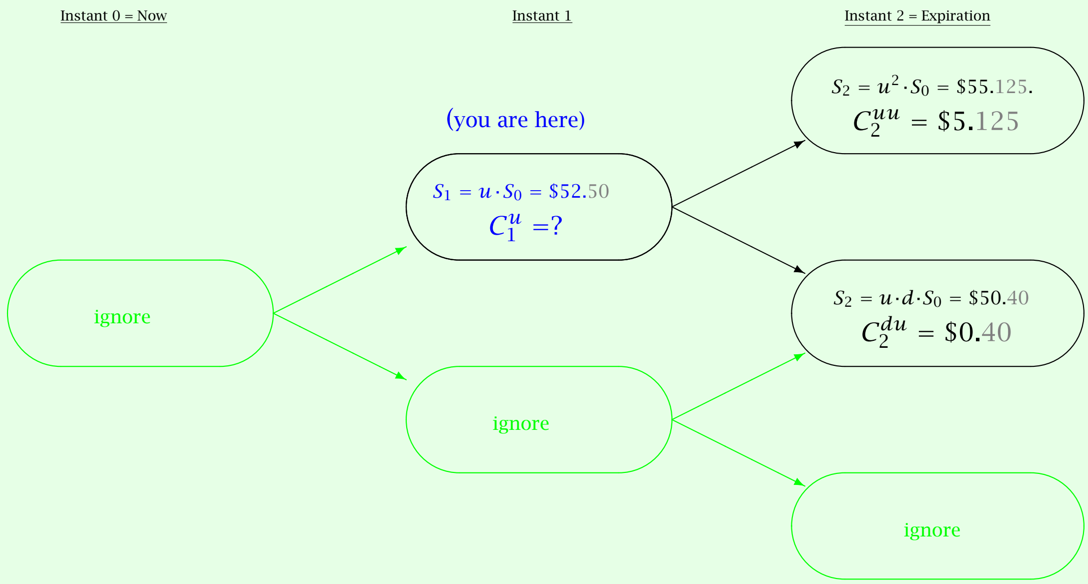
Simplifying the pricing with formulas.

$$\begin{aligned} \delta \cdot 0.96 \cdot \$48 + b \cdot (1.001) &= \$0.00 \\ \delta \cdot 1.05 \cdot \$48 + b \cdot (1.001) &= \$0.40 \\ \delta \cdot d \cdot S_0 + b \cdot (1+r) &= C_d \\ \delta \cdot u \cdot S_0 + b \cdot (1+r) &= C_u \quad . \end{aligned} \tag{I.35}$$

The solution is

$$\delta = \frac{\$0.40 - \$0.00}{1.05 \cdot \$48 - 0.96 \cdot \$48} \approx 0.0926 \quad , \quad b = -\$4.262 \quad . \tag{I.36}$$

So, if you purchase a portfolio of 0.0926 shares (costing $0.0926 \cdot \$48 \approx \$4.44$) and borrow \$4.262 (for a net outlay of \$0.182 today), then in the next period, this portfolio will pay off \$0 in the downstate and \$0.40 in the upstate. Because this is exactly the same as the payoff on the call option, the C_1^d call option should also be worth \$0.182.



Repeat the option pricing on the upper triangle.

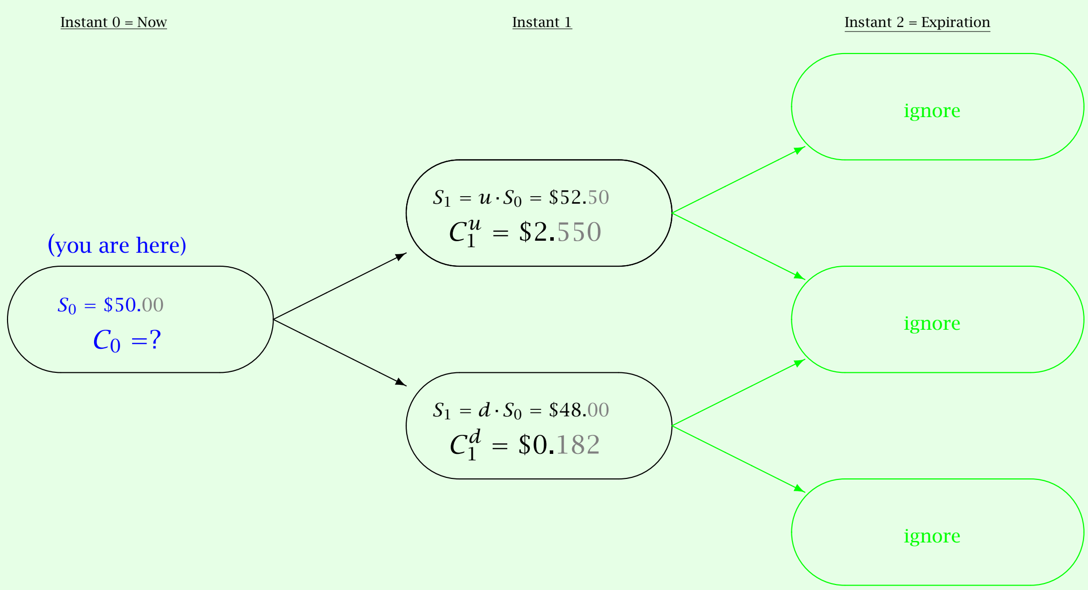
Now repeat the same exercise where the stock price stands at \$52.50, and next instant you can end up with either \$0.40 in the downstate or \$5.13 in the upstate. In this case, solve

$$\begin{aligned} \delta \cdot 0.96 \cdot \$52.50 + b \cdot (1.001) &= \$0.400 \\ \delta \cdot 1.05 \cdot \$52.50 + b \cdot (1.001) &= \$5.125 \\ \delta \cdot d \cdot S_0 + b \cdot (1+r) &= C_d \\ \delta \cdot u \cdot S_0 + b \cdot (1+r) &= C_u \quad , \end{aligned} \tag{I.37}$$

and the solutions are

$$\delta = \frac{\$5.125 - \$0.40}{1.05 \cdot \$52.50 - 0.96 \cdot \$52.50} \approx 1.00 \quad , \quad b = -\$49.95 \quad . \tag{I.38}$$

If you purchase 1.00 shares (at a price of \$52.50) and borrow \$49.95 (for a net portfolio cost of \$2.550), you will receive \$5.125 if the stock price goes up and \$0.40 if the stock price goes down. Therefore, after the stock price has gone up once to stand at \$52.50, the C_1^u call option has to be valued at \$2.550, too.



Repeat the option pricing on the left triangle.

To determine the value of the call C_0 at the outset, find the price of a security that will be worth \$0.182 if the stock moves from \$50 to \$48, and worth \$2.4975 if the stock moves from \$50 to \$52.50:

$$\begin{aligned}
\delta \cdot 0.96 \cdot \$50.00 + b \cdot (1.001) &= \$0.182 \\
\delta \cdot 1.05 \cdot \$50.00 + b \cdot (1.001) &= \$2.550 \\
\delta \cdot d \cdot S_0 + b \cdot (1+r) &= C_d \\
\delta \cdot u \cdot S_0 + b \cdot (1+r) &= C_u \quad .
\end{aligned} \tag{I.39}$$

The solution is

$$\delta = \frac{\$2.550 - \$0.182}{1.05 \cdot \$50 - 0.96 \cdot \$50} \approx 0.5262 \ , \quad b = \$25.04 \ . \tag{I.40}$$

You have to purchase 0.526 shares (cost today: \$26.31), and borrow \$25.04 dollars. Your portfolio's total net outlay is \$26.31 − \$25.04 ≈ \$1.27. Therefore, it follows that, by arbitrage, the price of the call option C_0 must be \$1.26 today.

Done!

The main idea behind the B-S formula (and most other option pricing techniques) has now been explained. The rest of this section only develops the algebra behind this idea a bit further.

Simplifying the binomial calculations.

In all these calculations, it was assumed that the stock can only move up by a factor of 1.05 or down by a factor of 0.96 in each of two instants. The algebra can be speeded up. Solve the two equations in (I.39) for δ and b, and you find that

$$\begin{aligned}
\delta &= \frac{C_u - C_d}{(u-d)\cdot S_0} = \left(\frac{\$2.55 - \$0.18}{1.05 - 0.96}\right) \cdot \$50 = 0.5267 \quad , \\
b &= \frac{d \cdot C_u - u \cdot C_d}{(1+r)\cdot(d-u)} = \frac{0.96 \cdot \$2.55 - 1.05 \cdot \$0.18}{(1+0.001)\cdot(0.96-1.05)} = -\$25.075 \quad .
\end{aligned} \tag{I.41}$$

The value of the call option (C) today is δ (the share investment) times the price of the stock, plus the price of the bond. So,

$$\begin{aligned}
C_0 &= \delta \cdot S_0 + b \\
&= \left[\frac{C_u - C_d}{(u-d)\cdot S_0}\right] \cdot S_0 + \left[\frac{d \cdot C_u - u \cdot C_d}{(1+r)\cdot(d-u)}\right] \\
&= \left[\left(\frac{1+r-d}{u-d}\right) \cdot C_u + \left(\frac{u-r-1}{u-d}\right) \cdot C_d\right] / (1+r) \\
&= \left[\left(\frac{1.001 - 0.96}{1.05 - 0.96}\right) \cdot \$2.55 + \left(\frac{1.05 - 1.001}{1.05 - 0.96}\right) \cdot \$0.18\right] / (1+0.001) \\
&= \$1.26 \quad .
\end{aligned} \tag{I.42}$$

We can further speed the computations up. Define p (often called the **risk-neutral probability**) as follows:

$$\begin{aligned}
p &\equiv \frac{1+r-d}{u-d} &\iff \quad (1-p) &\equiv \frac{u-r-1}{u-d} \\
p &\equiv \frac{1+0.001-0.96}{1.05-0.96} = 0.4555 &\iff \quad (1-p) &\equiv \frac{1.05-1.001}{1.05-0.96} = 0.5444 \quad .
\end{aligned} \tag{I.43}$$

We can now solve very quickly for the value of a binomial call option. First, compute p, and then use the formula

$$
\begin{aligned}
C_0 &= \frac{p \cdot C_u + (1-p) \cdot C_d}{1+r} \\
C_0 &= \frac{0.4555 \cdot \$2.55 + 0.5444 \cdot \$0.18}{1+0.001} = \$1.26 \; .
\end{aligned}
\tag{I.44}
$$

Truly convenient: Given a value of a contingent claim if *up* occurs vs. if *down* occurs, and given the interest rate, the calculations are now speedy. To determine the price of a contingent claim at any location in the tree, just substitute C_u and C_d into Formula I.44. It can be used to price all sorts of securities, not just call options. As long as you know how much your contingent claim is worth in either state, Formula I.44 tells you its value.

Pricing is risk-neutral, because pseudo-probabilities replace real probabilities.

Now, stare at Formula I.44 for a moment. The value of C looks almost like a discounted expected value. Yet, p is not a probability, but only a function of how much the price can move up or down each instant. In fact, whether the probability of an up-move each instant is 10% or 90% is irrelevant for the pricing of the option (although it does matter to the pricing of the stock). This fact—that the underlying true probability is not in the valuation formula—is also why the expected rate of return on the underlying asset does not enter the option valuation formula. It is simply irrelevant. The formula itself, however, looks so similar to risk-free discounting that it is just called **risk-neutral pricing**, because one computes the expected values with the pseudo-probability p (not the real probability) and then works out contingent claim prices as if the world were risk-neutral. The risk-neutral pricing method can value any security that depends only on the stock price.

A binomial is a very good B-S approximation, even with few levels.

The Black-Scholes formula is the limit of a binomial process in which there are infinitely many periods. But, even just a few levels in a binomial tree can often yield very good approximations to the B-S formula. For example, for a stock price of \$100, a risk-free rate of return defined by $e^r = 1 + 8\%$, a European option with 6 months to expiration and a volatility ($\sigma = 20\%$) would be valued as

	Levels in the Tree			
Strike Price K	10	20	50	∞ (B-S)
\$90	\$14.45	\$14.49	\$14.47	\$14.46
\$100	\$7.48	\$7.55	\$7.59	\$7.62
\$110	\$3.22	\$3.38	\$3.35	\$3.34

For further details, please consult an option pricing book.

Side Note: The binomial tree method has no problem with any types of derivative claims even if the stock pays dividends. At each tree node just before the stock goes ex-dividend, you must determine whether it is better to continue holding the call option on the stock (but with the stock reduced by the dividend) or whether it is better to exercise the call option in order to obtain the stock with its dividend payment. Then, you simply continue working backwards in the tree, assuming best behavior at each node.

Solutions and Exercises

1. They are similar in that both are bullish bets. However, they have very different payoff tables.
2. The long call option with a strike price of \$60 pays off above \$60, the long put option with a strike price of \$80 pays off below \$80:

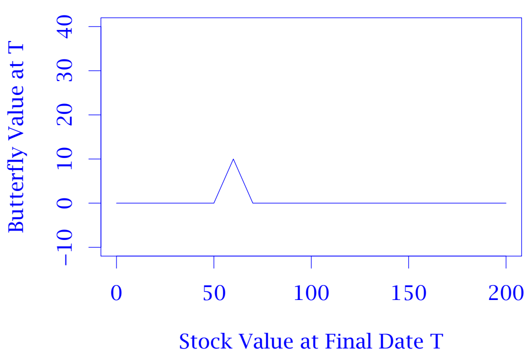
$\text{Stock}_{t=T}$	$\text{Pfio}_{t=T}$	$\text{Stock}_{t=T}$	$\text{Pfio}_{t=T}$
\$0	+\$80	\$70	+\$20
\$20	+\$60	\$75	+\$20
\$40	+\$40	\$80	+\$20
\$60	+\$20	\$90	+\$30
\$65	+\$20	\$100	+\$40

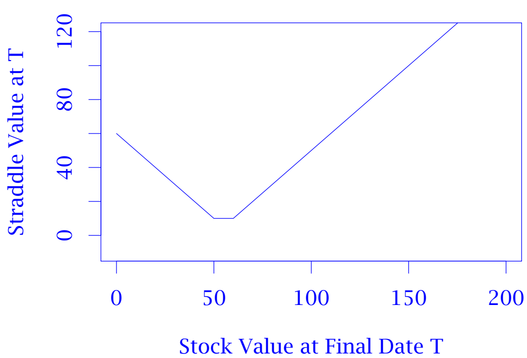
3. The short call option with a strike price of \$60 costs money above \$60, the short put option with a strike price of \$80 costs money below \$80:

$\text{Stock}_{t=T}$	$\text{Pfio}_{t=T}$	$\text{Stock}_{t=T}$	$\text{Pfio}_{t=T}$
\$0	–\$80	\$70	–\$20
\$20	–\$60	\$75	–\$20
\$40	–\$40	\$80	–\$20
\$60	–\$20	\$90	–\$30
\$65	–\$20	\$100	–\$40

4. The short call option with a strike price of \$60 costs money off above \$60, the long put option with a strike price of \$80 pays off below \$80:

$\text{Stock}_{t=T}$	$\text{Pfio}_{t=T}$	$\text{Stock}_{t=T}$	$\text{Pfio}_{t=T}$
\$0	+\$80	\$70	\$0
\$20	+\$60	\$75	–\$10
\$40	+\$40	\$80	–\$20
\$60	+\$20	\$90	–\$30
\$65	+\$10	\$100	–\$40

5. 

6. 

7. The formal definitions are on Page 9. In English: options cannot have negative values (i.e., prices); American options cannot earn arbitrage profits if they are purchased and exercised immediately; lower-strike price calls must be more expensive than higher-strike price calls (reverse for puts); and longer-term American options must be worth more than shorter-term American options.

8.

$$C_0 \;=\; P_0(K) + S_0 - \mathrm{PV}_0(K) \tag{I.45}$$

The price of a call option today, the price of the same put option (strike price and expiration time) today, the stock price, and the present value of the stock price.

9. put-call parity states that

$$\begin{aligned} C_0 &= P_0(K) + S_0 - \mathrm{PV}_0(K) \\ P_0 &= C_0(K) + \mathrm{PV}_0(K) - S_0 \;=\; \$20 + \$80/(1+10\%) - \$70 \;=\; \$22.73 \;. \end{aligned} \tag{I.46}$$

10. The put option should cost \$22.73, but it indeed costs \$25.00. Therefore, it is too expensive, and we definitely need to short it. To cover ourselves after shorting it, we now need to “manufacture” an artificial put option to neutralize our exposure. Put-call parity is

$$P_0 \;=\; C_0(K) + \mathrm{PV}_0(K) - S_0 \;=\; \$2.73 \;. \tag{I.47}$$

Loosely translated, a long put is a long call, a long present value of a strike price, and a short stock. Try purchasing one call (outflow today), saving the present value of the strike price (outflow), and shorting the stock (inflow today):

		Stock Price at Expiration T will be				
		$S_T < \$80$			$S_T > \$80$	
Execute	Today	\$60	\$70	\$80	\$90	\$100
Purchase 1 Call ($K = \$80$):	−\$20.00	\$0	\$0	\$0	\$10	\$20
Sell 1 Share:	+\$70.00	−\$60	−\$70	−\$80	−\$90	−\$100
Save to pay strike price:	−\$72.73	+\$80	+\$80	+\$80	+\$80	+\$80
Sell 1 put ($K = \$80$):	+\$25.00	−\$20	−\$10	\$0	\$0	\$0
Net	+\$2.27	\$0	\$0	\$0	\$0	\$0

You would earn an immediate arbitrage profit of \$2.27.

11. When there is no value to early exercise. This happens if the option is a call option on a stock that pays no dividends.

12. The (B-S) answer is that this is equivalent to owning the underlying stock itself. Therefore, $C_0 = S_0 = \$80.50$.

13. Here is a sample Excel Spreadsheet to compute B-S values:

	A	B	C	D	E
1	’S	’K	’t	’r	’s
2	80	90	0.25	0.05	0.20
3					
4	’PV(K)	88.90887	=B2/(1+D2)^C2		
5	’S*sqrt(T)	0.1	=E2*sqrt(C2)		
6	’xs	-1.00585	=ln(A2/B4)/B5+0.5*B5		
7	’xk	-1.10585	=B6-B5		
8	’B-S call	0.630532	=A2*NORMSDIST(B6) - B4*NORMSDIST(B7)		

In Excel, ’ln’ is the natural log; ’log’ is the log to the base of 10, which is the wrong function here. The NORMSDIST function is built into Excel, and computes $\mathcal{N}(\cdot)$.

14. $BS = \$6.89$. (Hint: $\mathcal{N}(0.817) \approx 0.793$, $\mathcal{N}(0.717) \approx 0.763$.)

15. BS(\$80,\$80,3/12,5%,20%)=\$3.68. (Hint: $\mathcal{N}(0.172) \approx 0.568$, $\mathcal{N}(0.072) \approx 0.529$.)

16. BS(\$80,\$75,3/12,5%,20%)=\$7.56. (Hint: $\mathcal{N}(0.934) \approx 0.825$, $\mathcal{N}(0.834) \approx 0.798$.)

17. BS(\$80,\$75,3/12,1.05,0.3)=\$8.15. (Hint: $\mathcal{N}(0.587) = 0.721$. $\mathcal{N}(0.437) = 0.669$.)

18. Use put-call parity. $P = \$0.98$.

19. \$7.87.

20. \$5.56+\$0.26 = \$6.89 + \$0.98 = \$5.82.

21. You can do so either with the BS formula, or through insight. The answer is \$5.

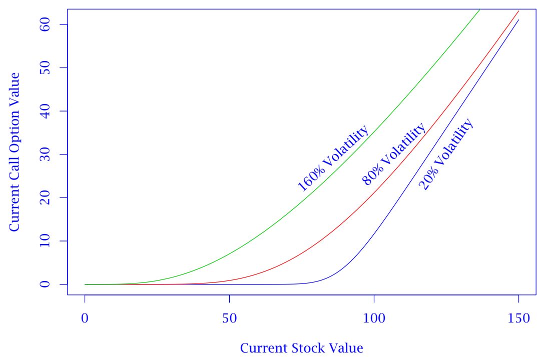
22. The B-S value $BS(\$80.50, \$100, {}^{34}/_{255}, 1.77\%, 30\%) = \$0.0930$. Therefore, the European put would be worth \$19.36.

23. If you hold onto it, you have an asset worth \$19.36. If you exercise it, you receive an immediate \$100−\$80.50 = \$19.50. Therefore, you would be better off exercising immediately!

24. It increases with the share price, length to maturity, volatility, and interest rate; and decreases with the strike price.

25.

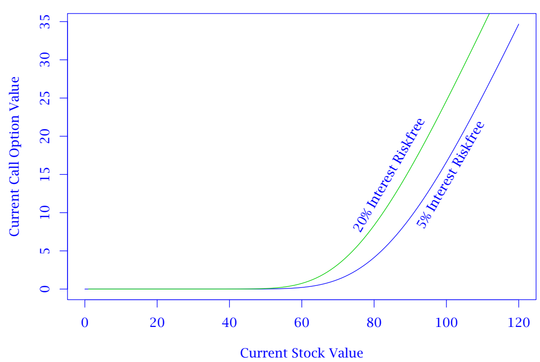
Figure:



Parameters: strike price $K = \$90$. Time to Expiration $t = 0.25$ years. Interest Rate: 5% per annum. Interpretation: The figure shows that the call option is more valuable when the volatility is higher. However, for very high stock prices, i.e., when the option is already far in-the-money, the advantage of volatility is relatively smaller.

26.

Figure:



Parameters: strike price $K = \$90$. Time to Expiration $t = 0.25$ years. Volatility: 20% per annum. Interpretation: The figure shows that the call option is more valuable when the interest rate is higher.

27. Do it!

28. Solve!

29. Assume that you must decide to produce at the start of this week. If you see that the price of gold is above \$400, then you extract gold. Otherwise, you do not. You can now value the gold mine as if it was 100 Black-Scholes options, each with current price \$365, strike price of \$400, interest rate of 10%, and volatility of 40% per annum.

30. It would be the sum of many such options. Each week, the production cost per ounce increases by about 0.35%. So, it would increase from \$400 to \$401.40, then to \$402.81, etc.

$$\begin{aligned}
\text{Value} \;=\; & \; BS(S = \$365, K = \$400, t = 1, r = 10\%, \sigma = 40\%) \\
& + BS(S = \$365, K = \$401.40, t = 2, r = 10\%, \sigma = 40\%) \\
& + BS(S = \$365, K = \$402.81, t = 3, r = 10\%, \sigma = 40\%) \; + \; \ldots
\end{aligned} \tag{I.48}$$

(All answers should be treated as suspect. They have only been sketched, and not been checked.)