2

Banach Spaces and Differential Calculus

Manifolds have enough structure to allow differentiation of maps between them. To set the stage for these concepts requires a development of differential calculus in linear spaces from a geometric point of view. The goal of this chapter is to provide this perspective.

Perhaps the most important theorem for later use is the *Implicit Function Theorem*. A fairly detailed exposition of this topic will be given with examples appropriate for use in manifold theory. The basic language of tangents, the derivative as a linear map, and the chain rule, while elementary, are important for developing geometric and analytic skills needed in manifold theory.

The main goal is to develop the theory of finite-dimensional manifolds. However, it is instructive and efficient to do the infinite-dimensional theory simultaneously. To avoid being sidetracked by infinite-dimensional technicalities at this stage, some functional analysis background and other topics special to the infinite-dimensional case are presented in supplements. With this arrangement readers who wish to concentrate on the finite-dimensional theory can do so with a minimum of distraction.

2.1 Banach Spaces

It is assumed the reader is familiar with the concept of a real or complex vector space. Banach spaces are vector spaces with the additional structure of a *norm* that defines a complete metric space. While most of this book is concerned with finite-dimensional spaces, much of the theory is really no harder in the general case, and the infinite-dimensional case is needed for certain applications. Thus, it makes sense to work in the setting of Banach spaces. In addition, although the primary concern is with *real* Banach spaces, the basic concepts needed for *complex* Banach spaces are introduced with little extra effort.

Normed Spaces. We begin with the notion of a normed space; that is, a space in which one has a length measure for vectors.

2.1.1 Definition. *A* ***norm*** *on a real (complex) vector space* $\mathbf{E}$ *is a mapping from* $\mathbf{E}$ *into the real numbers,* $\|\cdot\| : \mathbf{E} \to \mathbb{R}$*;* $e \mapsto \|e\|$*, such that*

N1. $\|e\| \geq 0$ *for all* $e \in \mathbf{E}$ *and* $\|e\| = 0$ *implies* $e = 0$ (positive definiteness);

N2. $\|\lambda e\| = |\lambda| \, \|e\|$ *for all* $e \in \mathbf{E}$ *and* $\lambda \in \mathbb{R}$ (homogeneity);

N3. $\|e_1 + e_2\| \leq \|e_1\| + \|e_2\|$ *for all* $e_1, e_2 \in \mathbf{E}$ (triangle inequality).

The pair $(\mathbf{E}, \|\cdot\|)$ *is sometimes called a* ***normed space****. If there is no danger of confusion, we sometimes just say "*$\mathbf{E}$ *is a normed space." To distinguish different norms, different notations are sometimes used, for example,*

$$\|\cdot\|_{\mathbf{E}}, \; \|\cdot\|_1, \; |||\cdot|||, \; etc.,$$

for the norm.

Example. Euclidean space $\mathbb{R}^n$ with the ***standard norm***

$$\|x\| = \left(x_1^2 + \cdots + x_n^2\right)^{1/2},$$

where $x = (x_1, \ldots, x_n)$, is a normed space. Proving that this norm satisfies the triangle inequality is probably easiest to do using properties of the inner product, which are considered below. Another norm on the same space is given by

$$|||x||| = \sum_{i=1}^{n} |x_i|,$$

as may be verified directly.

The triangle inequality **N3** has the following important consequence:

$$|\,\|e_1\| - \|e_2\|\,| \leq \|e_1 - e_2\| \quad \text{for all } e_1, e_2 \in \mathbf{E},$$

which is proved in the following way:

$$\begin{aligned} \|e_2\| &= \|e_1 + (e_2 - e_1)\| \leq \|e_1\| + \|e_1 - e_2\|, \\ \|e_1\| &= \|e_2 + (e_1 - e_2)\| \leq \|e_2\| + \|e_1 - e_2\|, \end{aligned}$$

so that both $\|e_2\| - \|e_1\|$ and $\|e_1\| - \|e_2\|$ are smaller than or equal to $\|e_1 - e_2\|$.

Seminormed Spaces. If **N1** in Definition 2.1.1 is replaced by

N1′. $\|e\| \geq 0$ *for all* $e \in \mathbf{E}$,

the mapping $\|\cdot\| : \mathbf{E} \to \mathbb{R}$ is called a ***seminorm***. For example, the function defined on $\mathbb{R}^2$ by $\|(x, y)\| = |x|$ is a seminorm.

Inner Product Spaces. These are spaces in which, roughly speaking, one can measure angles between vectors as well as their lengths.

2.1.2 Definition. *An* ***inner product*** *on a real vector space* $\mathbf{E}$ *is a mapping* $\langle \cdot, \cdot \rangle : \mathbf{E} \times \mathbf{E} \to \mathbb{R}$*, which we denote* $(e_1, e_2) \mapsto \langle e_1, e_2 \rangle$ *such that*

I1. $\langle e, e_1 + e_2 \rangle = \langle e, e_1 \rangle + \langle e, e_2 \rangle$;

I2. $\langle e, \alpha e_1 \rangle = \alpha \langle e, e_1 \rangle$;

I3. $\langle e_1, e_2 \rangle = \langle e_2, e_1 \rangle$;

I4. $\langle e, e \rangle \geq 0$ *and* $\langle e, e \rangle = 0$ *iff* $e = 0$.

The ***standard inner product*** on $\mathbb{R}^n$ is

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i,$$

and **I1–I4** are readily checked.

For vector spaces over the complex numbers, the definition is modified slightly as follows.

2.1.2′ Definition. A ***complex inner product*** or a ***Hermitian inner product*** on a complex vector space $\mathbf{E}$ is a mapping

$$\langle \cdot, \cdot \rangle : \mathbf{E} \times \mathbf{E} \to \mathbb{C}$$

such that the following conditions hold:

CI1. $\langle e, e_1 + e_2 \rangle = \langle e, e_1 \rangle + \langle e, e_2 \rangle$;

CI2. $\langle \alpha e, e_1 \rangle = \alpha \langle e, e_1 \rangle$;

CI3. $\langle e_1, e_2 \rangle = \overline{\langle e_2, e_1 \rangle}$ (so $\langle e, e \rangle$ is real);

CI4. $\langle e, e \rangle \geq 0$ and $\langle e, e \rangle = 0$ iff $e = 0$.

These properties are to hold for all $e, e_1, e_2 \in \mathbf{E}$ and $\alpha \in \mathbb{C}$; $\overline{z}$ denotes the complex conjugate of the complex number z. Note that **CI2** and **CI3** imply that $\langle e_1, \alpha e_2 \rangle = \overline{\alpha} \langle e_1, e_2 \rangle$. Properties **CI1–CI3** are also known in the literature under the name ***sesquilinearity***. As is customary, for a complex number z we shall denote by

$$\operatorname{Re} z = \frac{z + \overline{z}}{2}, \quad \operatorname{Im} z = \frac{z - \overline{z}}{2i}, \quad |z| = (z\overline{z})^{1/2}$$

its real and imaginary parts and its absolute value. The ***standard inner product*** on the product space $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ is defined by

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i \overline{w}_i,$$

and **CI1–CI4** are readily checked. Also $\mathbb{C}^n$ is a normed space with

$$\|z\|^2 = \sum_{i=1}^{n} |z_i|^2.$$

In $\mathbb{R}^n$ or $\mathbb{C}^n$, property **N3** is a little harder to check directly. However, as we shall show in Proposition 2.1.4, **N3** follows from **I1–I4** or **CI1–CI4**.

In a (real or complex) inner product space $\mathbf{E}$, two vectors $e_1, e_2 \in \mathbf{E}$ are called ***orthogonal*** and we write $e_1 \perp e_2$ provided $\langle e_1, e_2 \rangle = 0$. For a subset $A \subset \mathbf{E}$, the set $A^\perp$ defined by

$$A^\perp = \{ e \in \mathbf{E} \mid \langle e, x \rangle = 0 \text{ for all } x \in A \}$$

is called the ***orthogonal complement*** of A. Two sets $A, B \subset \mathbf{E}$ are called ***orthogonal*** and we write $A \perp B$ if $\langle A, B \rangle = 0$; that is, $e_1 \perp e_2$ for all $e_1 \in A$ and $e_2 \in B$.

Cauchy–Schwartz Inequality. This inequality will be a critical way to estimate inner products in terms of lengths.

2.1.3 Theorem (Cauchy–Schwartz Inequality). *In a (real or complex) inner product space,*

$$|\langle e_1, e_2 \rangle| \leq \langle e_1, e_1 \rangle^{1/2} \langle e_2, e_2 \rangle^{1/2}.$$

Equality holds iff e_1, e_2 are linearly dependent.

Proof. It suffices to prove the complex case. If $\alpha, \beta \in \mathbb{C}$, then

$$0 \leq \langle \alpha e_1 + \beta e_2, \alpha e_1 + \beta e_2 \rangle = |\alpha|^2 \langle e_1, e_1 \rangle + 2 \operatorname{Re}(\alpha \overline{\beta} \langle e_1, e_2 \rangle) + |\beta|^2 \langle e_2, e_2 \rangle .$$

If we set $\alpha = \langle e_2, e_2 \rangle$, and $\beta = -\langle e_1, e_2 \rangle$, then this becomes

$$0 \le \langle e_2, e_2 \rangle^2 \langle e_1, e_1 \rangle - 2 \langle e_2, e_2 \rangle |\langle e_1, e_2 \rangle|^2 + |\langle e_1, e_2 \rangle|^2 \langle e_2, e_2 \rangle ,$$

and so

$$\langle e_2, e_2 \rangle |\langle e_1, e_2 \rangle|^2 \le \langle e_2, e_2 \rangle^2 \langle e_1, e_1 \rangle .$$

If $e_2 = 0$, equality results in the statement of the proposition and there is nothing to prove. If $e_2 \neq 0$, the term $\langle e_2, e_2 \rangle$ in the preceding inequality can be cancelled since $\langle e_2, e_2 \rangle \neq 0$ by **CI4**. Taking square roots yields the statement of the proposition. Finally, equality results if and only if $\alpha e_1 + \beta e_2 = \langle e_2, e_2 \rangle e_1 - \langle e_1, e_2 \rangle e_2 = 0$. ■

2.1.4 Proposition. *Let $(\mathbf{E}, \langle \cdot, \cdot \rangle)$ be a (real or complex) inner product space and set $\|e\| = \langle e, e \rangle^{1/2}$. Then $(\mathbf{E}, \| \cdot \|)$ is a normed space.*

Proof. **N1** and **N2** are straightforward verifications. As for **N3**, the Cauchy–Schwartz inequality and the obvious inequality

$$\operatorname{Re}(\langle e_1, e_2 \rangle) \le |\langle e_1, e_2 \rangle|$$

imply

$$\begin{aligned}\|e_1 + e_2\|^2 &= \|e_1\|^2 + 2 \operatorname{Re}(\langle e_1, e_2 \rangle) + \|e_2\|^2 \le \|e_1\|^2 + 2|\langle e_1, e_2 \rangle| + \|e_2\|^2 \\ &\le \|e_1\|^2 + 2\|e_1\| \, \|e_2\| + \|e_2\|^2 = (\|e_1\| + \|e_2\|)^2\end{aligned}$$

■

Polarization and the Parallelogram Law. Some other useful facts about inner products are given next.

2.1.5 Proposition. *Let $(\mathbf{E}, \langle \cdot, \cdot \rangle)$ be an inner product space and $\| \cdot \|$ the corresponding norm. Then*

(i) (Polarization)

$$4 \langle e_1, e_2 \rangle = \|e_1 + e_2\|^2 - \|e_1 - e_2\|^2.$$

for $\mathbf{E}$ real, while

$$4 \langle e_1, e_2 \rangle = \|e_1 + e_2\|^2 - \|e_1 - e_2\|^2 + i\|e_1 + ie_2\|^2 - i\|e_1 - ie_2\|^2,$$

if $\mathbf{E}$ is complex.

(ii) (Parallelogram law)

$$2\|e_1\|^2 + 2\|e_2\|^2 = \|e_1 + e_2\|^2 + \|e_1 - e_2\|^2.$$

Proof. (i) In the complex case, we manipulate the right-hand side as follows

$$\begin{aligned}&\|e_1 + e_2\|^2 - \|e_1 - e_2\|^2 + i\|e_1 + ie_2\|^2 - i\|e_1 - ie_2\|^2 \\ &\quad = \|e_1\|^2 + 2 \operatorname{Re}(\langle e_1, e_2 \rangle) + \|e_2\|^2 \\ &\qquad - \|e_1\|^2 + 2 \operatorname{Re}(\langle e_1, e_2 \rangle) - \|e_2\|^2 \\ &\qquad + i\|e_1\|^2 + 2i \operatorname{Re}(\langle e_1, ie_2 \rangle) + i\|e_2\|^2 \\ &\qquad - i\|e_1\|^2 + 2i \operatorname{Re}(\langle e_1, ie_2 \rangle) - i\|e_2\|^2 \\ &\quad = 4 \operatorname{Re}(\langle e_1, e_2 \rangle) + 4i \operatorname{Re}(-i \langle e_1, e_2 \rangle) \\ &\quad = 4 \operatorname{Re}(\langle e_1, e_2 \rangle) + 4i \operatorname{Im}(\langle e_1, e_2 \rangle) \\ &\quad = 4 \langle e_1, e_2 \rangle .\end{aligned}$$

The real case is proved in a similar way.

(ii) We manipulate the right hand side:

$$\begin{aligned}\|e_1+e_2\|^2+\|e_1-e_2\|^2 &= \|e_1\|^2+2\,\mathrm{Re}(\langle e_1,e_2\rangle)+\|e_2\|^2+\|e_1\|^2\\ &\quad -2\,\mathrm{Re}(\langle e_1,e_2\rangle)+\|e_2\|^2\\ &= 2\|e_1\|^2+2\|e_2\|^2\end{aligned}$$ ■

Not all norms come from an inner product. For example, the norm

$$|||x||| = \sum_{i=1}^{n} |x_i|$$

is not induced by any inner product since this norm fails to satisfy the parallelogram law (see Exercise 2.1-1 for a discussion).

Normed Spaces are Metric Spaces. We have seen that inner product spaces are normed spaces. Now we show that normed spaces are metric spaces.

2.1.6 Proposition. *Let* $(\mathbf{E}, \|\cdot\|)$ *be a normed (resp. a seminormed) space and define* $d(e_1,e_2)=\|e_1-e_2\|$*. Then* $(\mathbf{E}, d)$ *is a metric (resp. pseudometric) space.*

Proof. The only non-obvious verification is the triangle inequality for the metric. By **N3**, we have

$$\begin{aligned}d(e_1,e_3) = \|e_1-e_3\| = \|(e_1-e_2)+(e_2-e_3)\| &\le \|e_1-e_2\|+\|e_2-e_3\|\\ = d(e_1,e_2)+d(e_2,e_3).\end{aligned}$$ ■

Thus we have the following hierarchy of generality:

More General →

inner product spaces	⊂	normed spaces	⊂	metric spaces	⊂	topological spaces

← More Special

Since inner product and normed spaces are metric spaces, we can use the concepts from Chapter 1. In a normed space, **N1** and **N2** imply that the maps $(e_1,e_2)\mapsto e_1+e_2$, $(\alpha,e)\mapsto \alpha e$ of $\mathbf{E}\times\mathbf{E}\to\mathbf{E}$, and $\mathbb{C}\times\mathbf{E}\to\mathbf{E}$, respectively, are continuous. Hence for $e_0\in\mathbf{E}$, and $\alpha_0\in\mathbb{C}$ ($\alpha_0\neq 0$) fixed, the mappings $e\mapsto e_0+e$, $e\mapsto\alpha_0 e$ are homeomorphisms. Thus, U is a neighborhood of the origin iff $e+U=\{\,e+x \mid x\in U\,\}$ is a neighborhood of $e\in\mathbf{E}$. In other words, all the neighborhoods of $e\in\mathbf{E}$ are sets that contain translates of disks centered at the origin. This constitutes a complete description of the topology of a normed vector space $(\mathbf{E},\|\cdot\|)$.

Finally, note that the inequality $|\|e_1\|-\|e_2\||\le\|e_1-e_2\|$ implies that the norm is uniformly continuous on $\mathbf{E}$. In inner product spaces, the Cauchy–Schwartz inequality implies the continuity of the inner product as a function of two variables.

Banach and Hilbert Spaces. Now we are ready to add the crucial assumption of completeness.

2.1.7 Definition. *Let* $(\mathbf{E},\|\cdot\|)$ *be a normed space. If the corresponding metric* d *is complete, we say* $(\mathbf{E},\|\cdot\|)$ *is a* ***Banach space****. If* $(\mathbf{E},\langle\cdot,\cdot\rangle)$ *is an inner product space whose corresponding metric is complete, we say* $(\mathbf{E},\langle\cdot,\cdot\rangle)$ *is a* ***Hilbert space****.*

For example, it is proven in books on advanced calculus that $\mathbb{R}^n$ is complete. Thus, $\mathbb{R}^n$ with the standard norm is a Banach space and with the standard inner product is a Hilbert space. Not only is the standard norm on $\mathbb{R}^n$ complete, but so is the nonstandard one

$$|||x||| = \sum_{i=1}^{n} |x_i|.$$

However, it is equivalent to the standard one in the following sense.

2.1.8 Definition. *Two norms on a vector space $\mathbf{E}$ are **equivalent** if they induce the same topology on $\mathbf{E}$.*

2.1.9 Proposition. *Two norms $\|\cdot\|$ and $|||\cdot|||$ on $\mathbf{E}$ are equivalent iff there is a constant M such that, for all $e \in \mathbf{E}$,*

$$\frac{1}{M}|||e||| \leq \|e\| \leq M|||e|||.$$

Proof. Let

$$B_r^1(x) = \{\, y \in \mathbf{E} \mid \|y - x\| \leq r \,\}, \quad B_r^2(x) = \{\, y \in \mathbf{E} \mid |||y - x||| \leq r \,\}$$

denote the two closed disks of radius r centered at $x \in \mathbf{E}$ in the two metrics defined by the norms $\|\cdot\|$ and $|||\cdot|||$, respectively. Since neighborhoods of an arbitrary point are translates of neighborhoods of the origin, the two topologies are the same iff for every $R > 0$, there are constants $M_1, M_2 > 0$ such that

$$B_{M_1}^2(0) \subset B_R^1(0) \subset B_{M_2}^2(0).$$

The first inclusion says that if $|||x||| \leq M_1$, then $\|x\| \leq R$, that is, if $|||x||| \leq 1$, then $\|x\| \leq R/M_1$. Thus, if $e \neq 0$, then

$$\left\| \frac{e}{|||e|||} \right\| = \frac{\|e\|}{|||e|||} \leq \frac{R}{M_1},$$

that is, $\|e\| \leq (R/M_1)|||e|||$ for all $e \in \mathbf{E}$. Similarly, the second inclusion is equivalent to the assertion that $(M_2/R)|||e||| \leq \|e\|$ for all $e \in \mathbf{E}$. Thus the two topologies are the same if there exist constants $N_1 > 0$, $N_2 > 0$ such that

$$N_1|||e||| \leq \|e\| \leq N_2|||e|||$$

for all $e \in \mathbf{E}$. Taking $M = \max(N_2, 1/N_1)$ gives the statement of the proposition. ■

Products of Normed Spaces. If $\mathbf{E}$ and $\mathbf{F}$ are normed vector spaces, the map

$$\|\cdot\| : \mathbf{E} \times \mathbf{F} \to \mathbb{R}$$

defined by

$$\|(e, e')\| = \|e\| + \|e'\|$$

is a norm on $\mathbf{E} \times \mathbf{F}$ inducing the product topology. Equivalent norms on $\mathbf{E} \times \mathbf{F}$ are

$$(e, e') \mapsto \max(\|e\|, \|e'\|) \quad \text{and} \quad (e, e') \mapsto \left(\|e\|^2 + \|e'\|^2\right)^{1/2}.$$

The normed vector space $\mathbf{E} \times \mathbf{F}$ is usually denoted by $\mathbf{E} \oplus \mathbf{F}$ and called the ***direct sum*** of $\mathbf{E}$ and $\mathbf{F}$. Note that $\mathbf{E} \oplus \mathbf{F}$ is a Banach space iff both $\mathbf{E}$ and $\mathbf{F}$ are. These statements are readily checked.

Finite Dimensional Spaces. In the finite dimensional case equivalence and completeness are automatic.

2.1.10 Proposition. *Let* $\mathbf{E}$ *be a finite-dimensional real or complex vector space. Then*

(i) *there is a norm on* $\mathbf{E}$;

(ii) *all norms on* $\mathbf{E}$ *are equivalent*;

(iii) *all norms on* $\mathbf{E}$ *are complete.*

Proof. Let $e_1, \ldots, e_n$ denote a basis of $\mathbf{E}$, where n is the dimension of $\mathbf{E}$.

(i) A norm on $\mathbf{E}$ is given, for example, by

$$\|e\| = \sum_{i=1}^{n} |a^i|, \quad \text{where } e = \sum_{i=1}^{n} a^i e_i.$$

(ii) Let $\|\cdot\|'$ be any other norm on $\mathbf{E}$. If

$$e = \sum_{i=1}^{n} a^i e_i \quad \text{and} \quad f = \sum_{i=1}^{n} b^i e_i,$$

the inequality

$$\begin{aligned} |\,\|e\|' - \|f\|'| \leq \|e - f\|' &\leq \sum_{i=1}^{n} |a^i - b^i|\, \|e_i\|' \\ &\leq \max_{1 \leq i \leq n} \left\{ \|e_i\|' \right\} |||(a^1, \ldots, a^n) - (b^1, \ldots, b^n)||| \end{aligned}$$

shows that the map

$$(x^1, \ldots, x^n) \in \mathbb{C}^n \mapsto \left\| \sum_{i=1}^{n} x^i e_i \right\|' \in [0, \infty[$$

is continuous with respect to the $|||\cdot|||$-norm on $\mathbb{C}^n$ (Use $\mathbb{R}^n$ in the real case). Since the set $S = \{\, x \in \mathbb{C}^n \mid |||x||| = 1 \,\}$ is closed and bounded, it is compact. The restriction of this map to S is a continuous, strictly positive function, so it attains its minimum M_1 and maximum M_2 on S; that is,

$$0 < M_1 \leq \left\| \sum_{i=1}^{n} x^i e_i \right\|' \leq M_2$$

for all $(x_1, \ldots, x_n) \in \mathbb{C}^n$ such that $|||(x_1, \ldots, x_n)||| = 1$. Thus,

$$M_1 |||(x^1, \ldots, x^n)||| \leq \left\| \sum_{i=1}^{n} x^i e_i \right\|' \leq M_2 |||(x^1, \ldots, x^n)|||,$$

that is, $M_1 \|e\| \leq \|e\|' \leq M_2 \|e\|$, where

$$e = \sum_{i=1}^{n} x^i e_i.$$

Taking $M = \max(M_2, 1/M_1)$, Proposition 2.1.9 shows that $\|\cdot\|$ and $\|\cdot\|'$ are equivalent.

(iii) It is enough to observe that

$$(x^1, \ldots, x^n) \in \mathbb{C}^n \mapsto \sum_{i=1}^{n} x^i e_i \in \mathbf{E}$$

is a norm-preserving map (i.e., an isometry) between $(\mathbb{C}^n, |||\cdot|||)$ and $(\mathbf{E}, \|\cdot\|)$. ■

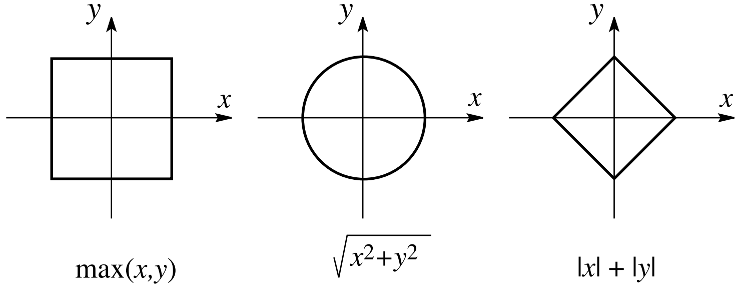

FIGURE 2.1.1. The unit spheres for various norms

The unit spheres for the three common norms on $\mathbb{R}^2$ are shown in Figure 2.1.1.

The foregoing proof shows that compactness of the unit sphere in a finite-dimensional space is crucial. This fact is exploited in the following supplement.

SUPPLEMENT 2.1A
A Characterization of Finite-Dimensional Spaces

2.1.11 Proposition. *A normed vector space is finite dimensional iff it is locally compact iff the closed unit disk is compact.*

Proof. If $\mathbf{E}$ is finite dimensional, the proof of Proposition 2.1.10(iii) shows that $\mathbf{E}$ is locally compact. Conversely, assume the closed unit disk $B_1(0) \subset \mathbf{E}$ is compact. Since it is compact, there is a finite covering of $B_1(0)$, by open discs of radius 1/2, say $\{ D_{1/2}(x_i) \mid i = 1, \ldots, n \}$. Let $\mathbf{F} = \operatorname{span}\{x_1, \ldots, x_n\}$. Since $\mathbf{F}$ is finite dimensional, it is homeomorphic to $\mathbb{C}^k$ (or $\mathbb{R}^k$) for some $k \le n$, and thus complete. Being a complete subspace of the metric space $(\mathbf{E}, \|\cdot\|)$, it is closed. We claim that $\mathbf{F} = \mathbf{E}$.

If not, there would exist $v \in \mathbf{E}$, $v \notin \mathbf{F}$. Since $\mathbf{F} = \operatorname{cl}(\mathbf{F})$, the number $d = \inf\{ \|v - e\| \mid e \in \mathbf{F} \}$ is strictly positive. Let $r > 0$ be such that $B_r(v) \cap \mathbf{F} \ne \varnothing$. The set $B_r(v) \cap \mathbf{F}$ is closed and bounded in the finite-dimensional space $\mathbf{F}$, so is compact. Since $\inf\{ \|v - e\| \mid e \in \mathbf{F} \} = \inf\{ \|v - e\| \mid e \in B_r(v) \subset \mathbf{F} \}$ and the continuous function defined by $e \in B_r(v) \cap \mathbf{F} \mapsto \|v - e\| \in]0, \infty[$ attains its minimum, there is a point $e_0 \in B_r(v) \cap \mathbf{F}$ such that $d = \|v - e_0\|$. But then there is a point x_i such that

$$\left\| \frac{v - e_0}{\|v - e_0\|} - x_i \right\| < \frac{1}{2},$$

so that

$$\|v - e_0 - (\|v - e_0\|)x_i\| < \frac{1}{2}\|v - e_0\| = \frac{d}{2}$$

Since $e_0 + \|v - e_0\|x_i \in \mathbf{F}$, we get $\|v - e_0 - (\|v - e_0\|)x_i\| \ge d$, which is a contradiction. ■

2.1.12 Examples.

A. Let X be a set and $\mathbf{F}$ a normed vector space. Define the set

$$B(X, \mathbf{F}) = \left\{ f : X \to \mathbf{F} \;\middle|\; \sup_{x \in X} \|f(x)\| < \infty \right\}.$$

Then $B(X, \mathbf{F})$ is easily seen to be a normed vector space with respect to the ***sup-norm***,

$$\|f\|_\infty = \sup_{x\in X} \|f(x)\|.$$

We prove that if $\mathbf{F}$ is complete, then $B(X, \mathbf{F})$ is a Banach space. Let $\{f_n\}$ be a Cauchy sequence in $B(X, \mathbf{F})$, that is,

$$\|f_n - f_m\|_\infty < \varepsilon \quad \text{for } n, m \geq N(\varepsilon).$$

Since for each $x \in X$, $\|f(x)\| \leq \|f\|_\infty$, it follows that $\{f_n(x)\}$ is a Cauchy sequence in $\mathbf{F}$, whose limit we denote by $f(x)$. In the inequality $\|f_n(x) - f_m(x)\| < \varepsilon$ for all $n, m \geq N(\varepsilon)$, let $m \to \infty$ and get $\|f_n(x) - f(x)\| \leq \varepsilon$ for $n \geq N(\varepsilon)$, that is, $\|f_n - f\|_\infty \leq \varepsilon$ for $n \geq N(\varepsilon)$. This shows that $f_n - f \in B(X, \mathbf{F})$, that is, that $f \in B(X, \mathbf{F})$, and that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. As a particular case, we get the Banach space c_b consisting of all bounded real sequences $\{a_n\}$ with the norm, also called the ***sup-norm***,

$$\|\{a_n\}\|_\infty = \sup_n |a_n|.$$

B. If X is a topological space, the space

$$CB(X, \mathbf{F}) = \{\, f : X \to \mathbf{F} \mid f \text{ is continuous, } f \in B(X, \mathbf{F}) \,\}$$

is closed in $B(X, \mathbf{F})$. Thus, if $\mathbf{F}$ is Banach, so is $CB(X, \mathbf{F})$. In particular, if X is a compact topological space and $\mathbf{F}$ is a Banach space, then

$$C(X, \mathbf{F}) = \{\, f : X \to \mathbf{F} \mid f \text{ continuous} \,\},$$

is a Banach space. For example, the vector space

$$C([0,1], \mathbb{R}) = \{\, f : [0,1] \to \mathbb{R} \mid f \text{ is continuous} \,\}$$

is a Banach space with the norm $\|f\|_\infty = \sup\{\, |f(x)| \mid x \in [0,1] \,\}$.

C. (For readers with some knowledge of measure theory.) Consider the space of real valued square integrable functions defined on an interval $[a, b] \subset \mathbb{R}$, that is, functions f that satisfy

$$\int_a^b |f(x)|^2 dx < \infty.$$

The function

$$\|\cdot\| : f \mapsto \left(\int_a^b |f(x)|^2 dx \right)^{1/2}$$

is, strictly speaking, not a norm on this space; for example, if

$$f(x) = \begin{cases} 0 & \text{for } x \neq a, \\ 1 & \text{for } x = a, \end{cases}$$

then $\|f\| = 0$, but $f \neq 0$. However, $\|\cdot\|$ does become a norm if we identify functions which differ only on a set of measure zero in $[a, b]$, that is, which are equal almost everywhere. The resulting vector space of equivalence classes $[f]$ will be denoted $L^2[a, b]$. With the norm of the equivalence class $[f]$ defined as

$$\|[f]\| = \left(\int_a^b |f(x)|^2 dx \right)^{1/2},$$

$L^2[a, b]$ is an (infinite-dimensional) Banach space. The only nontrivial part of this assertion is the completeness; this is proved in books on measure theory, such as Royden [1968]. As is customary, $[f]$ is denoted simply by f. In fact, $L^2[a, b]$ is a Hilbert space with

$$\langle f, g \rangle = \int_a^b f(x)g(x)\, dx.$$

If we use square integrable complex-valued functions we get a complex Hilbert space $L^2([a, b], \mathbb{C})$ with

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\, dx.$$

D. The space $L^p([a, b])$ may be defined for each real number $p \geq 1$ in an analogous fashion to $L^2[a, b]$. Functions $f : [a, b] \to \mathbb{R}$ satisfying

$$\int_a^b |f(x)|^p dx < \infty$$

are considered equivalent if they agree almost everywhere. $L^p([a, b])$ is then defined to be the vector space of equivalence classes of functions equal almost everywhere.

$$\| \cdot \|_p : L^p[a, b] \to \mathbb{R} \quad \text{given by} \quad [f] \to \left(\int_a^b |f(x)|^p dx \right)^{1/p}$$

defines a norm, called the L^p –norm, which makes $L^p[a, b]$ into an (infinite-dimensional) Banach space.

E. Denote by $C([a, b], \mathbb{R})$ the set of continuous real valued functions on $[a, b]$. With the L^1-norm, $C([a, b], \mathbb{R})$ is not a Banach space. For example, the sequence of continuous functions f_n shown in Figure 2.1.2 is a Cauchy sequence in the L^1-norm on $C([0, 1], \mathbb{R})$ but does not have a continuous limit function. On the other hand, with the sup norm

$$\|f\| = \sup_{x \in [0,1]} |f(x)|,$$

$C([0, 1])$ is complete, that is, it is a Banach space, as in Example B. ♦

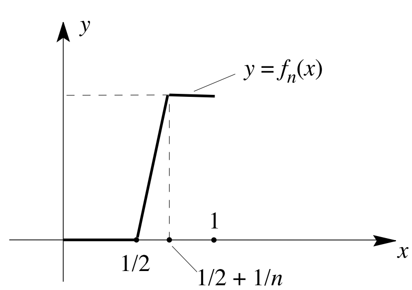

FIGURE 2.1.2. f_n converges in L_1, but not in C.

Quotients. As in the case of both topological spaces and vector spaces, quotient spaces of normed vector spaces play a fundamental role.

2.1.13 Proposition. *Let* $\mathbf{E}$ *be a normed vector space,* $\mathbf{F}$ *a closed subspace,* $\mathbf{E}/\mathbf{F}$ *the quotient vector space,*[1] *and* $\pi : \mathbf{E} \to \mathbf{E}/\mathbf{F}$ *the canonical projection defined by* $\pi(e) = [e] = e + \mathbf{F} \in \mathbf{E}/\mathbf{F}$.

(i) *The mapping* $\| \cdot \| : \mathbf{E}/\mathbf{F} \to \mathbb{R}$,

$$\|[e]\| = \inf\{ \|e + v\| \mid v \in \mathbf{F} \}$$

defines a norm on $\mathbf{E}/\mathbf{F}$.

(ii) π *is continuous and the topology on* $\mathbf{E}/\mathbf{F}$ *defined by the norm coincides with the quotient topology. In particular,* π *is open.*

(iii) *If* $\mathbf{E}$ *is a Banach space, so is* $\mathbf{E}/\mathbf{F}$.

Proof. (i) Clearly $\|[e]\| \geq 0$ for all $[e] \in \mathbf{E}/\mathbf{F}$ and

$$\|[0]\| = \inf\{ \|v\| \mid v \in \mathbf{F} \} = 0.$$

If $\|[e]\| = 0$, then there is a sequence $\{v_n\} \subset \mathbf{F}$ such that

$$\lim_{n\to\infty} \|e + v_n\| = 0.$$

Thus $\lim_{n\to\infty} v_n = -e$ and since $\mathbf{F}$ is closed, $e \in \mathbf{F}$; that is, $[e] = 0$. Thus **N1** is verified and the necessity of having $\mathbf{F}$ closed becomes apparent. **N2** and **N3** are straightforward verifications.

(ii) Since $\|[e]\| \leq \|e\|$, it is obvious that $\lim_{n\to\infty} e_n = e$ implies

$$\lim_{n\to\infty} \pi(e_n) = \lim_{n\to\infty} [e_n] = [e]$$

and hence π is continuous. Translation by a fixed vector is a homeomorphism. Thus to show that the topology of $\mathbf{E}/\mathbf{F}$ is the quotient topology, it suffices to show that if $[0] \in U$ and $\pi^{-1}(U)$ is a neighborhood of zero in $\mathbf{E}$, then U is a neighborhood of $[0]$ in $\mathbf{E}/\mathbf{F}$. Since $\pi^{-1}(U)$ is a neighborhood of zero in $\mathbf{E}$, there exists a disk $D_r(0) \subset \pi^{-1}(U)$. But then $\pi(D_r(0)) \subset U$ and $\pi(D_r(0)) = \{ [e] \mid e \in D_r(0) \} = \{ [e] \mid \|[e]\| < r \}$, so that U is a neighborhood of $[0]$ in $\mathbf{E}/\mathbf{F}$.

(iii) Let $\{[e_n]\}$ be a Cauchy sequence in $\mathbf{E}/\mathbf{F}$. We may assume without loss of generality that $\|[e_n] - [e_{n+1}]\| \leq 1/2^n$. Inductively, we find points $e'_n \in [e_n]$ such that $\|e'_n - e'_{n+1}\| < 1/2^n$. Thus $\{e'_n\}$ is Cauchy in $\mathbf{E}$ so it converges to, say, $e \in \mathbf{E}$. Continuity of π implies that $\lim_{n\to\infty}[e_n] = [e]$. ■

The ***codimension*** of $\mathbf{F}$ in $\mathbf{E}$ is defined to be the dimension of $\mathbf{E}/\mathbf{F}$. We say $\mathbf{F}$ is of finite codimension if $\mathbf{E}/\mathbf{F}$ is finite dimensional.

2.1.14 Definition. *The closed subspace* $\mathbf{F}$ *of the Banach space* $\mathbf{E}$ *is said to be* ***split****, or* ***complemented****, if there is a closed subspace* $\mathbf{G} \subset \mathbf{E}$ *such that* $\mathbf{E} = \mathbf{F} \oplus \mathbf{G}$.

The relation between split subspaces and quotients is simple: the projection map of $\mathbf{E}$ to $\mathbf{G}$ induces, in a natural way, a Banach space isomorphism of $\mathbf{E}/\mathbf{F}$ with $\mathbf{G}$. We leave this as a verification for the reader. One should note, however, that the quotient $\mathbf{E}/\mathbf{F}$ is defined independent of any choice of split subspace and that, accordingly, the choice of $\mathbf{G}$ is not unique.

[1] This quotient is the same as the quotient in the sense discussed in Chapter 1, with the equivalence relation being $u \sim v$ iff $u - v \in \mathbf{F}$, so that the equivalence class of u is the set $u + \mathbf{F}$.

Supplement 2.1B

Split Subspaces

Definition 2.1.14 implicitly asks that the topology of $\mathbf{E}$ coincide with the product topology of $\mathbf{F} \oplus \mathbf{G}$. We shall show in Supplement 2.2C that this topological condition can be dropped; that is, $\mathbf{F}$ is split iff $\mathbf{E}$ is the algebraic direct sum of $\mathbf{F}$ and the closed subspace $\mathbf{G}$.

As we noted above, if $\mathbf{E} = \mathbf{F} \oplus \mathbf{G}$ then $\mathbf{G}$ is isomorphic to $\mathbf{E}/\mathbf{F}$. However, $\mathbf{F}$ need not split for $\mathbf{E}/\mathbf{F}$ to be a Banach space, as we proved in Proposition 2.1.13. In finite-dimensional spaces, any subspace is closed and splits; however, in infinite dimensions this is false. For example, let $\mathbf{E} = L^p(S^1)$ and let

$$\mathbf{F} = \{ f \in \mathbf{E} \mid f(n) = 0 \text{ for } n < 0 \},$$

where

$$f(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta$$

is the nth Fourier coefficient of f. Then $\mathbf{F}$ is closed in $\mathbf{E}$, splits in $\mathbf{E}$ for $1 < p < \infty$ by a theorem of M. Riesz (Theorem 17.26 of Rudin [1966]) but does not split in $\mathbf{E}$ for $p = 1$ (Example 5.19 of Rudin [1973]). The same result holds if $\mathbf{E} = C^0(S^1, \mathbb{C})$ and $\mathbf{F}$ has the same definition.

Another example worth mentioning is $\mathbf{E} = \ell^\infty$, the Banach space of all bounded sequences, and $\mathbf{F} = c_0$, the subspace of ℓ^∞ consisting of all sequences convergent to zero. The subspace $\mathbf{F} = c_0$ is closed in $\mathbf{E} = \ell^\infty$, but does not split. However, c_0 splits in any *separable* Banach space which contains it isomorphically as a closed subspace by a theorem of Sobczyk; see Veech [1971]. If *every* subspace of a Banach space is complemented, the space must be isomorphic to a Hilbert space by a result of Lindenstrauss and Tzafriri [1971]. Supplement 2.2B gives some general criteria useful in nonlinear analysis for a subspace to be split. But the simplest situation occurs in Hilbert spaces.

2.1.15 Proposition. *If $\mathbf{E}$ is a Hilbert space and $\mathbf{F}$ a closed subspace, then $\mathbf{E} = \mathbf{F} \oplus \mathbf{F}^\perp$. Thus every closed subspace of a Hilbert space splits.*

The proof of this theorem is done in three steps, the first two being important results in their own rights.

2.1.16 Theorem (Minimal Norm Elements in Closed Convex Sets)**.** *If C is a closed convex set in $\mathbf{E}$, that is, $x, y \in C$ and $0 \le t \le 1$ implies*

$$tx + (1-t)y \in C,$$

then there exists a unique $e_0 \in C$ such that

$$\|e_0\| = \inf\{ \|e\| \mid e \in C \}.$$

Proof. Let $\sqrt{d} = \inf\{ \|e\| \mid e \in C \}$. Then there exists a sequence $\{e_n\}$ satisfying the inequality $d \le \|e_n\|^2 < d + 1/n$; hence $\|e_n\|^2 \to d$. Since $(e_n + e_m)/2 \in C$, C being convex, it follows that $\|(e_n+e_m)/2\|^2 \ge d$. By the parallelogram law,

$$\begin{aligned} \left\| \frac{e_n - e_m}{2} \right\|^2 &= 2\left\| \frac{e_n}{2} \right\|^2 + 2\left\| \frac{e_m}{2} \right\|^2 - \left\| \frac{e_n + e_m}{2} \right\|^2 \\ &< \frac{d}{2} + \frac{1}{2n} + \frac{d}{2} + \frac{1}{2m} - d = \frac{1}{2}\left(\frac{1}{n} + \frac{1}{m} \right); \end{aligned}$$

that is, $\{e_n\}$ is a Cauchy sequence in $\mathbf{E}$. Let $\lim_{n\to\infty} e_n = e_0$. Continuity of the norm implies that $\sqrt{d} = \lim_{n\to\infty} \|e_n\| = \|e_0\|$, and so the existence of an element of minimum norm in C is proved.

Finally, if f_0 is such that $\|e_0\| = \|f_0\| = \sqrt{d}$, the parallelogram law implies

$$\left\|\frac{e_0 - f_0}{2}\right\|^2 = 2\left\|\frac{e_0}{2}\right\| + 2\left\|\frac{f_0}{2}\right\|^2 - \left\|\frac{e_0 + f_0}{2}\right\|^2 \leq \frac{d}{2} + \frac{d}{2} - d = 0;$$

that is, $e_0 = f_0$. ■

2.1.17 Lemma. *Let $\mathbf{F} \subset \mathbf{E}$, $\mathbf{F} \neq \mathbf{E}$ be a closed subspace of $\mathbf{E}$. Then there exists a nonzero element $e_0 \in \mathbf{E}$ such that $e_0 \perp \mathbf{F}$.*

Proof. Let $e \in \mathbf{E}$, $e \notin \mathbf{F}$. The set $e - \mathbf{F} = \{ e - v \mid v \in \mathbf{F} \}$ is convex and closed, so by the previous lemma it contains a unique element $e_0 = e - v_0 \in e - \mathbf{F}$ of minimum norm. Since $\mathbf{F}$ is closed and $e \notin \mathbf{F}$, it follows that $e_0 \neq 0$. We shall prove that $e_0 \perp \mathbf{F}$.

Since e_0 is of minimal norm in $e - \mathbf{F}$, for any $v \in \mathbf{F}$ and $\lambda \in \mathbb{C}$ (resp., $\mathbb{R}$), we have

$$\|e_0\| = \|e - v_0\| \leq \|e - v_0 + \lambda v\| = \|e_0 + \lambda v\|,$$

that is, $2\,\mathrm{Re}(\lambda \langle v, e_0\rangle) + |\lambda|^2 \|v\|^2 \geq 0$.

If $\lambda = a \langle e_0, v\rangle$, $a \in \mathbb{R}$, $a \neq 0$, this becomes

$$a|\langle v, e_0\rangle|^2 (2 + a\|v\|^2) \geq 0$$

for all $v \in \mathbf{F}$, and $a \in \mathbb{R}$, $a \neq 0$. This forces $\langle v, e_0\rangle = 0$ for all $v \in \mathbf{F}$, since if $-2/\|v\|^2 < a < 0$, the preceding expression is negative. ■

Proof of Proposition 2.1.15. It is easy to see that $\mathbf{F}^\perp$ is closed (Exercise 2.1-3). We now show that $\mathbf{F} \oplus \mathbf{F}^\perp$ is a closed subspace. If

$$\{e_n + e'_n\} \subset \mathbf{F} \oplus \mathbf{F}^\perp, \quad \{e_n\} \subset \mathbf{F}, \quad \{e'_n\} \subset \mathbf{F}^\perp,$$

the relation

$$\|(e_n + e'_n) - (e_m + e'_m)\|^2 = \|e_n - e'_n\|^2 + \|e_m - e'_m\|^2$$

shows that $\{e_n + e'_n\}$ is Cauchy iff both $\{e_n\} \subset \mathbf{F}$ and $\{e'_n\} \subset \mathbf{F}^\perp$ are Cauchy. Thus if $\{e_n + e'_n\}$ converges, then there exist $e \in \mathbf{F}$, $e' \in \mathbf{F}^\perp$ such that $\lim_{n\to\infty} e_n = e$, $\lim_{n\to\infty} e'_n = e'$. Thus

$$\lim_{n\to\infty} (e_n + e'_n) = e + e' \in \mathbf{F} \oplus \mathbf{F}^\perp.$$

If $\mathbf{F} \oplus \mathbf{F}^\perp \neq \mathbf{E}$, then by the previous lemma there exists $e_0 \in \mathbf{E}$, $e_0 \notin \mathbf{F} \oplus \mathbf{F}^\perp$, $e_0 \neq 0$, $e_0 \perp (\mathbf{F} \oplus \mathbf{F}^\perp)$. Hence $e_0 \in \mathbf{F}^\perp$ and $e_0 \in \mathbf{F}$ so that $\langle e_0, e_0\rangle = \|e_0\|^2 = 0$; that is, $e_0 = 0$, a contradiction. ■

Exercises

◇ **2.1-1.** Show that a normed space is an inner product space iff the norm satisfies the parallelogram law. Conclude that if $n \geq 2$, $|||x||| = \sum |x_i|$ on $\mathbb{R}^n$ does not arise from an inner product.
Hint: Use the polarization identities over $\mathbb{R}$ and $\mathbb{C}$ to guess the corresponding inner-products.

◇ **2.1-2.** Let c_0 be the space of real sequences $\{a_n\}$ such that $a_n \to 0$ as $n \to \infty$. Show that c_0 is a closed subspace of the space c_b of bounded sequences (see Example 2.1.12A) and conclude that c_0 is a Banach space.

◇ **2.1-3.** Let $\mathbf{E}_1$ be the set of all C^1 functions $f : [0,1] \to \mathbb{R}$ with the norm

$$\|f\| = \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)|.$$

(i) Prove that $\mathbf{E}_1$ is a Banach space.

(ii) Let $\mathbf{E}_0$ be the space of C^0 maps $f : [0,1] \to \mathbb{R}$, as in Example 2.1.12. Show that the inclusion map $\mathbf{E}_1 \to \mathbf{E}_0$ is compact; that is, the unit ball in $\mathbf{E}_1$ has compact closure $\mathbf{E}_0$.

HINT: Use the Arzela–Ascoli theorem.

◇ **2.1-4.** Let $(\mathbf{E}, \langle\cdot,\cdot\rangle)$ be an inner product space and A, B subsets of $\mathbf{E}$. Define the ***sum*** of A and B by $A + B = \{\, a + b \mid a \in A,\ b \in B \,\}$. Show that:

(i) $A \subset B$ implies $B^\perp \subset A^\perp$;

(ii) $A^\perp$ is a closed subspace of $\mathbf{E}$;

(iii) $A^\perp = (\mathrm{cl}(\mathrm{span}(A)))^\perp$, $(A^\perp)^\perp = \mathrm{cl}(\mathrm{span}(A))$;

(iv) $(A + B)^\perp = A^\perp \cap B^\perp$; and

(v) $(\mathrm{cl}(\mathrm{span}(A)) \cap \mathrm{cl}(\mathrm{span}(B)))^\perp = A^\perp + B^\perp$ (*not* necessarily a direct sum).

◇ **2.1-5.** A sequence $\{e_n\} \subset \mathbf{E}$, where $\mathbf{E}$ is an inner product space, is said to be ***weakly convergent*** to $e \in \mathbf{E}$ iff all the numerical sequences $\langle v, e_n\rangle$ converge to $\langle v, e\rangle$ for all $v \in \mathbf{E}$. Let

$$\ell^2(\mathbb{C}) = \left\{ \{a_n\} \;\middle|\; a_n \in \mathbb{C} \text{ and } \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$$

and put

$$\langle \{a_n\}, \{b_n\} \rangle = \sum_{n=1}^{\infty} a_n \overline{b_n}.$$

Show that:

(i) in any inner product space, convergence implies weak convergence;

(ii) $\ell^2(\mathbb{C})$ is an inner product space;

(iii) the sequence $(1,0,0,\dots), (0,1,0,\dots), (0,0,1,\dots), \dots$ is not convergent but is weakly convergent to 0 in $\ell^2(\mathbb{C})$.

NOTE: $\ell^2(\mathbb{C})$ is in fact complete, so it is a Hilbert space. The ambitious reader can attempt a direct proof or consult a book on real analysis such as Royden [1968].

◇ **2.1-6.** Show that a normed vector space is a Banach space iff every absolutely convergent series is convergent. (A series $\sum_{n=1}^{\infty} x_n$ is called ***absolutely convergent*** if $\sum_{n=1}^{\infty} \|x_n\|$ converges.)

◇ **2.1-7.** Let $\mathbf{E}$ be a Banach space and $\mathbf{F}_1 \subset \mathbf{F}_2 \subset \mathbf{E}$ be closed subspaces such that $\mathbf{F}_2$ splits in $\mathbf{E}$. Show that $\mathbf{F}_1$ splits in $\mathbf{E}$ iff $\mathbf{F}_1$ splits in $\mathbf{F}_2$.

◇ **2.1-8.** Let $\mathbf{F}$ be closed in $\mathbf{E}$ of finite codimension. Show that if $\mathbf{G}$ is a subspace of $\mathbf{E}$ containing $\mathbf{F}$, then $\mathbf{G}$ is closed.

◇ **2.1-9.** Let $\mathbf{E}$ be a Hilbert space. A set $\{e_i\}_{i\in I}$ is called ***orthonormal*** if $\langle e_i, e_j\rangle = \delta_{ij}$, the Kronecker delta. An orthonormal set $\{e_i\}_{i\in I}$ is a ***Hilbert basis*** if $\mathrm{cl}(\mathrm{span}\{e_i\}_{i\in I}) = \mathbf{E}$.

(i) Let $\{e_i\}_{i\in I}$ be an orthonormal set and $\{e_{i(1)}, \ldots, e_{i(n)}\}$ be any finite subset. Show that

$$\sum_{j=1}^{n} \left|\left\langle e, e_{i(j)}\right\rangle\right|^2 \leq \|e\|^2$$

for any $e \in \mathbf{E}$.

HINT:

$$e' = e - \sum_{j=1,\ldots,n} \left\langle e, e_{i(j)}\right\rangle e_{i(j)}$$

is orthogonal to all $\{\, e_{i(j)} \mid j = 1, \ldots, n \,\}$.

(ii) Deduce from (i) that for any positive integer n, the set $\{\, i \in I \mid |\langle e, e_i\rangle| > 1/n \,\}$ has at most $n\|e\|^2$ elements. Hence at most countably many $i \in I$ satisfy $\langle e, e_i\rangle \neq 0$, for any $e \in \mathbf{E}$.

(iii) Show that any Hilbert space has a Hilbert basis.

HINT: Use Zorn's lemma and Lemma 2.1.17.

(iv) If $\{e_i\}_{i\in I}$ is a Hilbert basis in $\mathbf{E}$, $e \in \mathbf{E}$, and $\{e_{i(j)}\}$ is the (at most countable) set such that $\left\langle e, e_{i(j)}\right\rangle \neq 0$, show that

$$\sum_{j=1}^{\infty} \left|\left\langle e, e_{i(j)}\right\rangle\right|^2 = \|e\|^2.$$

HINT: If

$$e' = \sum_{j=1,\ldots,\infty} \left\langle e, e_{i(j)}\right\rangle e_{i(j)},$$

show that

$$\langle e_i, e - e'\rangle = 0 \quad \text{for all } i \in I$$

and then use maximality of $\{e_i\}_{i\in I}$.

(v) Show that $\mathbf{E}$ is separable iff any Hilbert basis is at most countable.

HINT: For the "if" part, show that the set

$$\left\{ \sum_{k=1}^{n} \alpha_n e_n \;\middle|\; \alpha_k = a_k + ib_k, \text{ where } a_k \text{ and } b_k \text{ are rational} \right\}$$

is dense in $\mathbf{E}$. For the "only if" part, show that since $\|e_i - e_j\|^2 = 2$, the disks of radius $1/\sqrt{2}$ centered at e_i are all disjoint.)

(vi) If $\mathbf{E}$ is a separable Hilbert space, it is algebraically isomorphic either with $\mathbb{C}^n$ or $\ell^2(\mathbb{C})$ ($\mathbb{R}^n$ or $\ell^2(\mathbb{R})$), and the algebraic isomorphism can be chosen to be norm preserving.

2.2 Linear and Multilinear Mappings

This section deals with various aspects of linear and multilinear maps between Banach spaces. We begin with a study of continuity and go on to study spaces of continuous linear and multilinear maps and some related fundamental theorems of linear analysis.

Continuity and Boundedness. We begin by showing for a linear map, the equivalence of continuity and possessing a certain bound.

2.2.1 Proposition. *Let $A : \mathbf{E} \to \mathbf{F}$ be a linear map of normed spaces. Then A is continuous if and only if there is a constant $M > 0$ such that*

$$\|Ae\|_{\mathbf{F}} \leq M\|e\|_{\mathbf{E}} \quad \textit{for all } e \in \mathbf{E}.$$

Proof. Continuity of A at $e_0 \in \mathbf{E}$ means that for any $r > 0$, there exists $\rho > 0$ such that

$$A(e_0 + B_\rho(0_{\mathbf{E}})) \subset Ae_0 + B_r(0_{\mathbf{F}})$$

($0_{\mathbf{E}}$ denotes the zero element in $\mathbf{E}$ and $B_s(0_{\mathbf{E}})$ denotes the closed disk of radius s centered at the origin in $\mathbf{E}$). Since A is linear, this is equivalent to: if $\|e\|_{\mathbf{E}} \leq \rho$, then $\|Ae\|_{\mathbf{F}} \leq r$. If $M = r/\rho$, continuity of A is thus equivalent to the following: $\|e\|_{\mathbf{E}} \leq 1$ implies $\|Ae\|_{\mathbf{F}} \leq M$, which in turn is the same as: there exists $M > 0$ such that $\|Ae\|_{\mathbf{F}} \leq M\|e\|_{\mathbf{E}}$, which is seen by choosing the vector $e/\|e\|_{\mathbf{E}}$ in the preceding implication. ■

Because of this proposition one says that a continuous linear map is ***bounded***.

2.2.2 Proposition. *If $\mathbf{E}$ is finite dimensional and $A : \mathbf{E} \to \mathbf{F}$ is linear, then A is continuous.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis for $\mathbf{E}$. Letting

$$M_1 = \max(\|Ae_1\|, \ldots, \|Ae_n\|)$$

and setting $e = a^1 e_1 + \cdots + a^n e_n$, we see that

$$\begin{aligned} \|Ae\| &= \|a^1 Ae_1 + \cdots + a^n Ae_n\| \\ &\leq |a^1|\,\|Ae_1\| + \cdots + |a^n|\,\|Ae_n\| \leq M_1(|a^1| + \cdots + |a^n|). \end{aligned}$$

Since $\mathbf{E}$ is finite dimensional, all norms on it are equivalent. Since $|||e||| = \sum |a^i|$ is a norm, it follows that $|||e||| \leq C\|e\|$ for a constant C. Let $M = M_1 C$ and use Proposition 2.2.1. ■

Operator Norm. The bound on continuous linear maps suggests a norm for such maps.

2.2.3 Definition. *If $\mathbf{E}$ and $\mathbf{F}$ are normed spaces and $A : \mathbf{E} \to \mathbf{F}$ is a continuous linear map, let the **operator norm** of A be defined by*

$$\|A\| = \sup\left\{ \frac{\|Ae\|}{\|e\|} \;\middle|\; e \in \mathbf{E},\ e \neq 0 \right\}$$

(*which is finite by Proposition* 2.2.1). *Let $L(\mathbf{E}, \mathbf{F})$ denote the space of all continuous linear maps of $\mathbf{E}$ to $\mathbf{F}$. If $\mathbf{F} = \mathbb{C}$ (resp., $\mathbb{R}$), then $L(\mathbf{E}, \mathbb{C})$ (resp., $L(\mathbf{E}, \mathbb{R})$) is denoted by $\mathbf{E}^*$ and is called the **complex** (resp., **real**) **dual space** of $\mathbf{E}$. (It will always be clear from the context whether $L(\mathbf{E}, \mathbf{F})$ or $\mathbf{E}^*$ means the real or complex linear maps or dual space; in most of the work later in this book it will mean the real case.)*

A straightforward verification gives the following equivalent definitions of $\|A\|$:

$$\begin{aligned} \|A\| &= \inf\{\, M > 0 \mid \|Ae\| \leq M\|e\| \text{ for all } e \in \mathbf{E} \,\} \\ &= \sup\{\, \|Ae\| \mid \|e\| \leq 1 \,\} = \sup\{\, \|Ae\| \mid \|e\| = 1 \,\}. \end{aligned}$$

In particular, $\|Ae\| \leq \|A\|\,\|e\|$.

If $A \in L(\mathbf{E}, \mathbf{F})$ and $B \in L(\mathbf{F}, \mathbf{G})$, where $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$ are normed spaces, then

$$\|(B \circ A)(e)\| = \|B(A(e))\| \leq \|B\|\,\|Ae\| \leq \|B\|\,\|A\|\,\|e\|,$$

and so

$$\|(B \circ A)\| \leq \|B\|\,\|A\|.$$

Equality does not hold in general. A simple example is obtained by choosing $\mathbf{E} = \mathbf{F} = \mathbf{G} = \mathbb{R}^2$, $A(x, y) = (x, 0)$, and $B(x, y) = (0, y)$, so that $B \circ A = 0$ and $\|A\| = \|B\| = 1$.

2.2.4 Proposition. *$L(\mathbf{E},\mathbf{F})$ with the norm just defined is a normed space. It is a Banach space if $\mathbf{F}$ is.*

Proof. Clearly $\|A\| \geq 0$ and $\|0\| = 0$. If $\|A\| = 0$, then for any $e \in \mathbf{E}$, $\|Ae\| \leq \|A\|\,\|e\| = 0$, so that $A = 0$ and thus **N1** (see Definition 2.1.1) is verified. **N2** and **N3** are also straightforward to check.

Now let $\mathbf{F}$ be a Banach space and $\{A_n\} \subset L(\mathbf{E},\mathbf{F})$ be a Cauchy sequence. Because of the inequality $\|A_n e - A_m e\| \leq \|A_n - A_m\|\,\|e\|$ for each $e \in \mathbf{E}$, the sequence $\{A_n e\}$ is Cauchy in $\mathbf{F}$ and hence is convergent. Let $Ae = \lim_{n\to\infty} A_n e$. This defines a map $A : \mathbf{E} \to \mathbf{F}$, which is evidently linear. It remains to be shown that A is continuous and $\|A_n - A\| \to 0$.

If $\varepsilon > 0$ is given, there exists a natural number $N(\varepsilon)$ such that for all $m, n \geq N(\varepsilon)$ we have $\|A_n - A_m\| < \varepsilon$. If $\|e\| \leq 1$, this implies

$$\|A_n e - A_m e\| < \varepsilon,$$

and now letting $m \to \infty$, it follows that $\|A_n e - Ae\| \leq \varepsilon$ for all e with $\|e\| \leq 1$. Thus $A_n - A \in L(\mathbf{E},\mathbf{F})$, hence $A \in L(\mathbf{E},\mathbf{F})$ and $\|A_n - A\| \leq \varepsilon$ for all $n \geq N(\varepsilon)$; that is, $\|A_n - A\| \to 0$. ■

If a sequence $\{A_n\}$ converges to A in $L(\mathbf{E},\mathbf{F})$ in the sense that

$$\|A_n - A\| \to 0, \quad \text{that is,} \quad \text{if } A_n \to A$$

in the norm topology, we say $A_n \to A$ ***in norm***. This phrase is necessary since other topologies on $L(\mathbf{E},\mathbf{F})$ are possible. For example, we say that $A_n \to A$ ***strongly*** if $A_n e \to Ae$ for each $e \in \mathbf{E}$. Since $\|A_n e - Ae\| \leq \|A_n - A\|\,\|e\|$, norm convergence implies strong convergence. The converse is false as the following example shows. Let

$$\mathbf{E} = \ell^2(\mathbb{R}) = \left\{ \{a_n\} \;\middle|\; \sum_{n=1}^{\infty} a_n^2 < \infty \right\}$$

with inner product

$$\langle \{a_n\}, \{b_n\} \rangle = \sum_{n=1}^{\infty} a_n b_n.$$

Let

$$e_n = (0, \ldots, 0, 1, 0, \ldots) \in \mathbf{E}, \quad \mathbf{F} = \mathbb{R}, \quad \text{and} \quad A_n = \langle e_n, \cdot \rangle \in L(\mathbf{E},\mathbf{F}),$$

where the 1 in e_n is in the n^{th} slot. The sequence $\{A_n\}$ is not Cauchy in the operator norm since $\|A_n - A_m\| = \sqrt{2}$, but if $e = \{a_m\}$, $A_n(e) = \langle e_n, e \rangle = a_n \to 0$, that is, $A_n \to 0$ strongly. If both $\mathbf{E}$ and $\mathbf{F}$ are finite dimensional, strong convergence implies norm convergence. (To see this, choose a basis $e_1, \ldots, e_n$ of $\mathbf{E}$ and note that strong convergence is equivalent to $A_k e_i \to A e_i$ as $k \to \infty$ for $i = 1, \ldots, n$. Hence $\max_i \|Ae_i\| = |||A|||$ is a norm yielding strong convergence. But all norms are equivalent in finite dimensions.)

Supplement 2.2A
Dual Spaces

Riesz Representation Theorem. Recall from elementary linear algebra that the dual space of a finite dimensional vector space of dimension n also has dimension n and so the space and its dual are isomorphic. For general Banach spaces this is no longer true. However, it is true for Hilbert space.

2.2.5 Theorem (Riesz Representation Theorem). *Let* $\mathbf{E}$ *be a real (resp., complex) Hilbert space. The map* $e \mapsto \langle \cdot, e \rangle$ *is a linear (resp., antilinear) norm-preserving isomorphism of* $\mathbf{E}$ *with* $\mathbf{E}^*$*; for short,* $\mathbf{E} \cong \mathbf{E}^*$*. (A map* $A : \mathbf{E} \to \mathbf{F}$ *between complex vector spaces is called* ***antilinear*** *if we have the identities* $A(e + e') = Ae + Ae'$*, and* $A(\alpha e) = \overline{\alpha} Ae$*.)*

Proof. Let $f_e = \langle \cdot, e \rangle$. Then $\|f_e\| = \|e\|$ and thus $f_e \in \mathbf{E}^*$. The map $A : \mathbf{E} \to \mathbf{E}^*$, $Ae = f_e$ is clearly linear (resp. antilinear), norm preserving, and thus injective. It remains to prove surjectivity.

Let $f \in \mathbf{E}^*$ and $\ker(f) = \{ e \in \mathbf{E} \mid f(e) = 0 \}$. $\ker(f)$ is a closed subspace in $\mathbf{E}$. If $\ker(f) = \mathbf{E}$, then $f = 0$ and $f = A(0)$ so there is nothing to prove. If $\ker(f) \neq \mathbf{E}$, then by Lemma 2.1.17 there exists $e \neq 0$ such that $e \perp \ker(f)$. Then we claim that $f = A(f(e)e/\|e\|^2)$. Indeed, any $v \in \mathbf{E}$ can be written as

$$v = v - \frac{f(v)}{f(e)}e + \frac{f(v)}{f(e)}e \quad \text{and} \quad v - \frac{f(v)}{f(e)}e \in \ker(f). \qquad \blacksquare$$

Thus, in a real Hilbert space $\mathbf{E}$ every continuous linear function $\ell : \mathbf{E} \to \mathbb{R}$ can be written

$$\ell(e) = \langle e_0, e \rangle$$

for some $e_0 \in \mathbf{E}$ and $\|\ell\| = \|e_0\|$.

In a general Banach space $\mathbf{E}$ we do not have such a concrete realization of $\mathbf{E}^*$. However, one should *not* always attempt to identify $\mathbf{E}$ and $\mathbf{E}^*$, even in finite dimensions. In fact, distinguishing these spaces is fundamental in tensor analysis.

Reflexive Spaces. We have a canonical map $i : \mathbf{E} \to \mathbf{E}^{**}$ defined by

$$i(e)(\ell) = \ell(e).$$

Pause and look again at this strange but natural formula: $i(e) \in \mathbf{E}^{**} = (\mathbf{E}^*)^*$, so $i(e)$ is applied to the element $\ell \in \mathbf{E}^*$. It is easy to check that i is norm preserving. One calls $\mathbf{E}$ ***reflexive*** if i is onto. Hilbert spaces are reflexive, by Theorem 2.2.5. For example, let $V = L^2(\mathbb{R}^n)$ with inner product

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x) g(x)\, dx,$$

and let $\alpha : L^2(\mathbb{R}^n) \to \mathbb{R}$ be a continuous linear functional. Then the Riesz representation theorem guarantees that there exists a unique $g \in L^2(\mathbb{R}^n)$ such that

$$\alpha(f) = \int_{\mathbb{R}^n} g(x) f(x)\, dx = \langle g, f \rangle$$

for all $f \in L^2(\mathbb{R}^n)$.

In general, if $\mathbf{E}$ is not a Hilbert space and we wish to represent a linear functional α in the form of $\alpha(f) = \langle g, f \rangle$, we must regard $g(x)$ as an element of the dual space $\mathbf{E}^*$. For example, let $\mathbf{E} = C_0(\Omega, \mathbb{R})$, where $\Omega \subset \mathbb{R}^n$. Each $x \in \Omega$ defines a linear functional $\mathbf{E}_x : C_0(\Omega, \mathbb{R}) \to \mathbb{R}$; $f \mapsto f(x)$. This linear functional cannot be represented in the form $\mathbf{E}_x(f) = \langle g, f \rangle$ and, indeed, is not continuous in the L^2 norm. Nevertheless, it is customary and useful to write such linear maps as if $\langle\ ,\ \rangle$ were the L^2 inner product. Thus one writes, symbolically,

$$\mathbf{E}_{x_0}(f) = \int_{\Omega} \delta(x - x_0) f(x)\, dx,$$

which defines the ***Dirac delta function*** at x_0; that is, $g(x) = \delta(x - x_0)$.

Linear Extension Theorem. Next we shall discuss integration of vector valued functions. We shall require the following.

2.2.6 Theorem (Linear Extension Theorem). *Let* $\mathbf{E}$, $\mathbf{F}$, *and* $\mathbf{G}$ *be normed vector spaces where*

(i) $\mathbf{F} \subset \mathbf{E}$;

(ii) $\mathbf{G}$ *is a Banach space*; *and*

(iii) $T \in L(\mathbf{F}, \mathbf{G})$.

Then the closure $\mathrm{cl}(\mathbf{F})$ *of* $\mathbf{F}$ *is a normed vector subspace of* $\mathbf{E}$ *and* T *can be uniquely extended to a map* $\mathcal{T} \in L(\mathrm{cl}(\mathbf{F}), \mathbf{G})$. *Moreover, we have the equality* $\|T\| = \|\mathcal{T}\|$.

Proof. The fact that $\mathrm{cl}(\mathbf{F})$ is a linear subspace of $\mathbf{E}$ is easily checked. Note that if $\mathcal{T}$ exists it is unique by continuity. Let us prove the existence of $\mathcal{T}$. If $e \in \mathrm{cl}(\mathbf{F})$, we can write $e = \lim_{n\to\infty} e_n$, where $e_n \in \mathbf{F}$, so that

$$\|Te_n - Te_m\| \leq \|T\|\,\|e_n - e_m\|,$$

which shows that the sequence $\{Te_n\}$ is Cauchy in the Banach space $\mathbf{G}$. Let $\mathcal{T}e = \lim_{n\to\infty} Te_n$. This limit is independent of the sequence $\{e_n\}$, for if $e = \lim e'_n$, then

$$\|Te_n - Te'_n\| \leq \|T\|(\|e_n - e\| + \|e - e'_n\|),$$

which proves that $\lim_{n\to\infty}(Te_n) = \lim_{n\to\infty}(Te'_n)$. It is simple to check the linearity of $\mathcal{T}$. Since $Te = \mathcal{T}e$ for $e \in \mathbf{F}$ (because $e = \lim_{n\to\infty} e$), $\mathcal{T}$ is an extension of T. Finally,

$$\|\mathcal{T}e\| = \left\|\lim_{n\to\infty}(Te_n)\right\| = \lim_{n\to\infty}\|Te_n\| \leq \|T\| \lim_{n\to\infty}\|e_n\| = \|T\|\,\|e\|$$

shows that $\mathcal{T} \in L(\mathrm{cl}(\mathbf{F}), \mathbf{G})$ and $\|\mathcal{T}\| \leq \|T\|$. The inequality $\|T\| \leq \|\mathcal{T}\|$ is obvious since $\mathcal{T}$ extends T. ∎

Integration of Banach Space Valued Functions. As an application of the preceding lemma we define a Banach space valued integral that will be of use later on. Fix the closed interval $[a, b] \subset \mathbb{R}$ and the Banach space $\mathbf{E}$. A map $f : [a, b] \to \mathbf{E}$ is called a ***step function*** if there exists a partition $a = t_0 < t_1 < \cdots < t_n = b$ such that f is constant on each interval $[t_i, t_{i+1}[$. Using the standard notion of a refinement of a partition, it is clear that the sum of two step functions and the scalar multiples of step functions are also step functions. Thus the set $\mathcal{S}([a, b], \mathbf{E})$ of step functions is a vector subspace of $B([a, b], \mathbf{E})$, the Banach space of all bounded functions (see Example 2.1.12). The integral of a step function f is defined by

$$\int_a^b f = \sum_{i=0}^{n}(t_{i+1} - t_i) f(t_i).$$

It is easily verified that this definition is independent of the partition. Also note that

$$\left\|\int_a^b f\right\| \leq \int_a^b \|f\| \leq (b - a)\,\|f\|_\infty,$$

where $\|f\|_\infty = \sup_{a \leq t \leq b} |f(t)|$; that is,

$$\int_a^b : \mathcal{S}([a, b], \mathbf{E}) \to \mathbf{E}$$

is continuous and linear. By the linear extension theorem, it extends to a continuous linear map

$$\int_a^b \in L(\mathrm{cl}(\mathcal{S}([a, b], \mathbf{E})), \mathbf{E}).$$

2.2.7 Definition. *The extended linear map $\int_a^b$ is called the **Cauchy–Bochner** integral.*

Note that

$$\left\| \int_a^b f \right\| \le \int_a^b \|f\| \le (b-a)\,\|f\|_\infty.$$

The usual properties of the integral such as

$$\int_a^b f = \int_a^c f + \int_c^b f \quad \text{and} \quad \int_a^b f = -\int_b^a f$$

are easily verified since they clearly hold for step functions.

The space $\mathrm{cl}(\mathcal{S}([a,b],\mathbf{E})$ contains enough interesting functions for our purposes, namely

$$C^0([a,b],\mathbf{E}) \subset \mathrm{cl}(\mathcal{S}([a,b],\mathbf{E})) \subset B([a,b],\mathbf{E}).$$

The first inclusion is proved in the following way. Since $[a,b]$ is compact, each $f \in C^0([a,b],\mathbf{E})$ is uniformly continuous. For $\varepsilon > 0$, let $\delta > 0$ be given by uniform continuity of f for $\varepsilon/2$. Then take a partition $a = t_0 < \cdots < t_n = b$ such that $|t_{i+1} - t_1| < \delta$ and define a step function g by $g|[t_i, t_{i+1}[= f(t_i)$. Then the ε-disk $D_\varepsilon(f)$ in $B([a,b],\mathbf{E})$ contains g.

Finally, note that if $\mathbf{E}$ and $\mathbf{F}$ are Banach spaces, $A \in L(\mathbf{E},\mathbf{F})$, and $f \in \mathrm{cl}(\mathcal{S}([a,b],\mathbf{E}))$, we have $A \circ f \in \mathrm{cl}(\mathcal{S}([a,b],\mathbf{F}))$ since

$$\|A \circ f_n - A \circ f\| \le \|A\|\,\|f_n - f\|_\infty,$$

where f_n are step functions in $\mathbf{E}$. Moreover,

$$\int_a^b A \circ f = A\left(\int_a^b f\right)$$

since this relation is obtained as the limit of the same (easily verified) relation for step functions. The reader versed in Riemann integration should notice that this integral for $\mathbf{E} = \mathbb{R}$ is less general than the Riemann integral; that is, the Riemann integral exists also for functions outside of $\mathrm{cl}(\mathcal{S}([a,b],\mathbb{R}))$. For purposes of this book, however, this integral will suffice.

Multilinear Mappings. If $\mathbf{E}_1, \ldots, \mathbf{E}_k$ and $\mathbf{F}$ are linear spaces, a map

$$A : \mathbf{E}_1 \times \cdots \times \mathbf{E}_k \to \mathbf{F}$$

is called *k-**multilinear*** if $A(e_1, \ldots, e_k)$ is linear in each argument separately. Linearity in the first argument means that

$$A(\lambda e_1 + \mu f_1, e_2, \ldots, e_k) = \lambda A(e_1, e_2, \ldots, e_k) + \mu A(f_1, e_2, \ldots, e_k).$$

We shall study multilinear mappings in detail in our study of tensors. They also come up in the study of differentiation, and we shall require a few facts about them for that purpose.

2.2.8 Definition. *The space of all continuous k-multilinear maps from $\mathbf{E}_1 \times \cdots \times \mathbf{E}_k$ to $\mathbf{F}$ is denoted $L(\mathbf{E}_1, \ldots, \mathbf{E}_k; \mathbf{F})$. If $\mathbf{E}_i = \mathbf{E}$, $1 \le i \le k$, this space is denoted $L^k(\mathbf{E},\mathbf{F})$.*

As in Definition 2.1.1, a k-multilinear map A is continuous if and only if there is an $M > 0$ such that

$$\|A(e_1, \ldots, e_k)\| \le M\|e_1\| \cdots \|e_k\|$$

for all $e_i \in \mathbf{E}_i$, $1 \le i \le k$. We set

$$\|A\| = \sup\left\{ \frac{\|A(e_1, \ldots, e_k)\|}{\|e_1\| \cdots \|e_k\|} \;\middle|\; e_1, \ldots, e_k \neq 0 \right\},$$

which makes $L(\mathbf{E}_1, \ldots, \mathbf{E}_k; \mathbf{F})$ into a normed space that is complete if $\mathbf{F}$ is. Again $\|A\|$ can also be defined as

$$\begin{aligned}\|A\| &= \inf\{ M > 0 \mid \|A(e_1, \ldots, e_n)\| \le M\|e_1\| \cdots \|e_n\| \} \\ &= \sup\{ \|A(e_1, \ldots, e_n)\| \mid \|e_1\| \le 1, \ldots, \|e_n\| \le 1 \} \\ &= \sup\{ \|A(e_1, \ldots, e_n)\| \mid \|e_1\| = \cdots = \|e_n\| = 1 \}.\end{aligned}$$

2.2.9 Proposition. *There are (natural) norm-preserving isomorphisms*

$$\begin{aligned}L(\mathbf{E}_1, L(\mathbf{E}_2, \ldots, \mathbf{E}_k; \mathbf{F})) &\cong L(\mathbf{E}_1, \ldots, \mathbf{E}_k; \mathbf{F}) \\ &\cong L(\mathbf{E}_1, \ldots, \mathbf{E}_{k-1}; L(\mathbf{E}_k, \mathbf{F})) \\ &\cong L(\mathbf{E}_{i_1}, \ldots, \mathbf{E}_{i_k}; \mathbf{F})\end{aligned}$$

where $(i_1, \ldots, i_k)$ *is a permutation of* $(1, \ldots, k)$.

Proof. For $A \in L(\mathbf{E}_1, L(\mathbf{E}_2, \ldots, \mathbf{E}_k; \mathbf{F}))$, define $A' \in L(\mathbf{E}_1, \ldots, \mathbf{E}_k; \mathbf{F})$ by

$$A'(e_1, \ldots, e_k) = A(e_1)(e_2, \ldots, e_k).$$

The association $A \mapsto A'$ is clearly linear and $\|A'\| = \|A\|$. The other isomorphisms are proved similarly. ■

In a similar way, we can identify $L(\mathbb{R}, \mathbf{F})$ (or $L(\mathbb{C}, \mathbf{F})$ if $\mathbf{F}$ is complex) with $\mathbf{F}$: to $A \in L(\mathbb{R}, \mathbf{F})$ we associate $A(1) \in \mathbf{F}$; again $\|A\| = \|A(1)\|$. As a special case of Proposition 2.2.9 note that $L(\mathbf{E}, \mathbf{E}^*) \cong L^2(\mathbf{E}, \mathbb{R})$ (or $L^2(\mathbf{E}; \mathbb{C})$, if $\mathbf{E}$ is complex). This isomorphism will be useful when we consider second derivatives.

Permutations. We shall need a few facts about the permutation group on k elements. The information we cite is obtainable from virtually any elementary algebra book. The ***permutation group*** on k elements, denoted S_k, consists of all bijections $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$ together with the structure of a group under composition. Clearly, S_k has order $k!$, that is, S_k has $k!$ elements.

One of the more subtle but very useful properties of permutations is the notion of the *sign of a permutation*. Letting $(\mathbb{R}, \times)$ denote $\mathbb{R}\backslash\{0\}$ with the multiplicative group structure, the ***sign*** is a homomorphism

$$\text{sign} : S_k \to (\mathbb{R}, \times).$$

Being a ***homomorphism*** means that for $\sigma, \tau \in S_k$,

$$\text{sign}(\sigma \circ \tau) = (\text{sign } \sigma)(\text{sign } \tau).$$

The image of "sign" is the subgroup $\{-1, 1\}$, while its kernel consists of the subgroup of ***even permutations***. Thus, a permutation σ is ***even*** when sign $\sigma = +1$ and is ***odd*** when sign $\sigma = -1$.

The sign of a permutation is perhaps easiest to understand in terms of transpositions. A ***transposition*** is a permutation that swaps two elements of $\{1, \ldots, k\}$, leaving the remainder fixed. An even (odd) permutation can be written as the product of an even (odd) number of transpositions.

The group S_k ***acts*** on the space $L^k(\mathbf{E}; \mathbf{F})$; that is, each $\sigma \in S_k$ defines a map $\sigma : L^k(\mathbf{E}; \mathbf{F}) \to L^k(\mathbf{E}; \mathbf{F})$ by

$$(\sigma A)(e_1, \ldots, e_k) = A(e_{\sigma(1)}, \ldots, e_{\sigma(k)}).$$

Note that $(\tau\sigma)A = \tau(\sigma A)$ for all $\tau, \sigma \in S_k$. Accordingly, $A \in L^k(\mathbf{E}, \mathbf{F})$ is called ***symmetric*** (***antisymmetric***) if for any permutation $\sigma \in S_k$, $\sigma A = A$ (resp., $\sigma A = (\text{sign}\,\sigma)A$.)

2.2.10 Definition. *Let* $\mathbf{E}$ *and* $\mathbf{F}$ *be normed vector spaces. Let* $L_s^k(\mathbf{E};\mathbf{F})$ *and* $L_a^k(\mathbf{E};\mathbf{F})$ *denote the subspaces of symmetric and antisymmetric elements of* $L^k(\mathbf{E};\mathbf{F})$. *Write* $S^0(\mathbf{E},\mathbf{F}) = \mathbf{F}$ *and*

$$S^k(\mathbf{E},\mathbf{F}) = \{\, p : \mathbf{E} \to \mathbf{F} \mid p(e) = A(e,\ldots,e) \text{ for some } A \in L^k(\mathbf{E};\mathbf{F}) \,\}.$$

We call $S^k(\mathbf{E},\mathbf{F})$ *the* ***space of homogeneous polynomials of degree*** k *from* $\mathbf{E}$ *to* $\mathbf{F}$.

Note that $L_s^k(\mathbf{E};\mathbf{F})$ and $L_a^k(\mathbf{E};\mathbf{F})$ are closed in $L^k(\mathbf{E};\mathbf{F})$; thus if $\mathbf{F}$ is a Banach space, so are $L_s^k(\mathbf{E};\mathbf{F})$ and $L_a^k(\mathbf{E};\mathbf{F})$. The antisymmetric maps $L_a^k(\mathbf{E};\mathbf{F})$ will be studied in detail in Chapter 7. For technical purposes later in this chapter we will need a few facts about $S^k(\mathbf{E},\mathbf{F})$ which are given in the following supplement.

Supplement 2.2B

Homogeneous Polynomials

2.2.11 Proposition.

(i) $S^k(\mathbf{E},\mathbf{F})$ *is a normed vector space with respect to the following norm:*

$$\begin{aligned}\|f\| &= \inf\{\, M > 0 \mid \|f(e)\| \le M\|e\|^k \,\} = \sup\{\, \|f(e)\| \mid \|e\| \le 1 \,\}\\ &= \sup\{\, \|f(e)\| \mid \|e\| = 1 \,\}.\end{aligned}$$

It is complete if $\mathbf{F}$ *is.*

(ii) *If* $f \in S^k(\mathbf{E},\mathbf{F})$ *and* $g \in S^n(\mathbf{F},\mathbf{G})$, *then* $g \circ f \in S^{kn}(\mathbf{E},\mathbf{G})$ *and* $\|g \circ f\| \le \|g\|\,\|f\|$.

(iii) (*Polarization.*) *The mapping* $\,' : L^k(\mathbf{E},\mathbf{F}) \to S^k(\mathbf{E},\mathbf{F})$ *defined by* $A'(e) = A(e,\ldots,e)$ *restricted to* $L_s^k(\mathbf{E};\mathbf{F})$ *has an inverse* $\grave{}: S^k(\mathbf{E},\mathbf{F}) \to L_s^k(\mathbf{E},\mathbf{F})$ *given by*

$$\grave{f}(e_1,\ldots,e_k) = \frac{1}{k!}\frac{\partial^k}{\partial t_1\cdots\partial t_k}\bigg|_{t=0} f(t_1e_1+\cdots+t_ke_k).$$

(*Note that* $f(t_1e_1+\cdots+t_ke_k)$ *is a polynomial in* $t_1,\ldots,t_k$, *so there is no problem in understanding what the derivatives on the right hand side mean.*)

(iv) *For* $A \in L^k(\mathbf{E},\mathbf{F})$, $\|A'\| \le \|A\| \le (k^k/k!)\|A'\|$, *which implies the maps* $'$ *and* $\grave{}$ *are continuous.*

Proof. (i) and (ii) are proved exactly as for $L(\mathbf{E},\mathbf{F}) = S^1(\mathbf{E},\mathbf{F})$.

(iii) For $A \in L_s^k(\mathbf{E};\mathbf{F})$ we have

$$\begin{aligned}&A'(t_1e_1+\cdots+t_ke_k)\\ &\quad= \sum_{a_1+\cdots+a_j=k}\frac{k!}{a_1!\cdots a_j!}\,t_1^{a_1}\cdots t_j^{a_j}\,A(e_1,\ldots,e_1,\ldots,e_j,\ldots,e_j),\end{aligned}$$

where each e_i appears a_i times, and

$$\frac{\partial^k}{\partial t_1\cdots\partial t_k}\bigg|_{t=0} t_1^{a_1}\cdots t_j^{a_j} = \begin{cases}1, & \text{if } k=j,\\ 0, & \text{if } k \ne j.\end{cases}$$

It follows that

$$A(e_1,\ldots,e_k) = \frac{1}{k!}\frac{\partial^k}{\partial t_1\cdots\partial t_k}A'(t_1e_1+\cdots+t_ke_k),$$

and for $j \neq k$,

$$\frac{\partial^j}{\partial t_1 \cdots \partial t_j} A'(t_1 e_1 + \cdots + t_k e_k) = 0$$

This means that $\grave{}(A') = A$ for any $A \in L_s^k(\mathbf{E}, \mathbf{F})$.

Conversely, if $f \in S^k(\mathbf{E}, \mathbf{F})$, then

$$\begin{aligned}(\grave{f})'(e) = \grave{f}(e, \ldots, e) &= \frac{1}{k!} \frac{\partial^k}{\partial t_1 \cdots \partial t_k}\bigg|_{t=0} f(t_1 e + \cdots + t_k e) \\ &= \frac{1}{k!} \frac{\partial^k}{\partial t_1 \cdots \partial t_k}\bigg|_{t=0} (t_1 + \cdots + t_k)^k f(e) = f(e).\end{aligned}$$

(iv) $\|A'(e)\| = \|A(e, \ldots, e)\| \leq \|A\| \, \|e\|^k$, so $\|A'\| \leq \|A\|$. To prove the other inequality, note that if $A \in L_s^k(\mathbf{E}; \mathbf{F})$, then

$$A(e_1, \ldots, e_k) = \frac{1}{k! 2^k} \sum \varepsilon_1 \cdots \varepsilon_k \, A'(\varepsilon_1 e_1 + \cdots + \varepsilon_k e_k),$$

where the sum is taken over all the 2^k possibilities $\varepsilon_1 = \pm 1, \ldots, \varepsilon_k = \pm 1$. Put $\|e_1\| = \cdots = \|e_k\| = 1$ and get

$$\begin{aligned}\|A'(\varepsilon_1 e_1 + \cdots + \varepsilon_k e_k)\| &\leq \|A'\| \, \|\varepsilon_1 e_1 + \cdots + \varepsilon_k e_k\|^k \\ &\leq \|A'\| (|\varepsilon_1| \, \|e_1\| + \cdots + |\varepsilon_k| \, \|e_k\|)^k = \|A'\| k^k,\end{aligned}$$

whence

$$\|A(e_1, \ldots, e_k)\| \leq \frac{k^k}{k!} \|A'\|,$$

that is,

$$\|A\| \leq \frac{k^k}{k!} \|A'\|. \qquad \blacksquare$$

Let $\mathbf{E} = \mathbb{R}^n$, $\mathbf{F} = \mathbb{R}$, and $e_1, \ldots, e_n$ be the standard basis in $\mathbb{R}^n$. For $f \in S^k(\mathbb{R}^n, \mathbb{R})$, set

$$c_{a_1 \cdots a_n} = f(e_1, \ldots, e_1, \ldots, e_n, \ldots, e_n),$$

where each e_i appears a_i times. If $e = t_1 e_1 + \cdots + t_n e_n$, the proof of (iii) shows that

$$f(e) = \grave{f}(e, \ldots, e) = \sum_{a_1 + \cdots + a_n = k} c_{a_1 \ldots a_n} t_1^{a_1} \cdots t_n^{a_n},$$

that is, f is a homogeneous polynomial of degree k in $t_1, \ldots, t_n$ in the usual algebraic sense.

The constant $k^k/k!$ in (iv) is the best possible, as the following example shows. Write elements of $\mathbb{R}^k$ as $\mathbf{x} = (x^1, \ldots, x^k)$ and introduce the norm

$$|||(x^1, \ldots, x^k)||| = |x^1| + \cdots + |x^k|.$$

Define $A \in L_s^k(\mathbb{R}^k, \mathbb{R})$ by

$$A(x_1, \ldots, x_k) = \frac{1}{k!} \sum x_{i_1}^1 \ldots x_{i_k}^k,$$

where $\mathbf{x}_i = (x_i^1, \ldots, x_i^k) \in \mathbb{R}^k$ and the sum is taken over all permutations of $\{1, \ldots, k\}$. It is easily verified that $\|A\| = 1/k!$ and $\|A'\| = 1/k^k$; that is, $\|A\| = (k^k/k!)\|A'\|$. Thus, except for $k = 1$, the isomorphism $'$ is not norm preserving. (This is a source of annoyance in the theory of formal power series and infinite-dimensional holomorphic mappings.)

Supplement 2.2C
The Three Pillars of Linear Analysis

The three fundamental theorems of linear analysis are the *Hahn–Banach theorem*, the *open mapping theorem*, and the *uniform boundedness principle*. See, for example, Banach [1932] and Riesz and Sz.-Nagy [1952] for further information. This supplement gives the classical proofs of these three fundamental theorems and derives some corollaries that will be used later. In finite dimensions these corollaries are all "obvious."

Hahn–Banach Theorem. This basic result guarantees a rich supply of continuous linear functionals.

2.2.12 Theorem (Hahn–Banach Theorem)**.** *Let $\mathbf{E}$ be a real or complex vector space, $\|\cdot\| : \mathbf{E} \to \mathbb{R}$ a seminorm, and $\mathbf{F} \subset \mathbf{E}$ a subspace. If $f \in \mathbf{F}^*$ satisfies $|f(e)| \le \|e\|$ for all $e \in \mathbf{F}$, then there exists a linear map $f' : \mathbf{E} \to \mathbb{R}$ (or $\mathbb{C}$) such that $f'|\mathbf{F} = f$ and $|f'(e)| \le \|e\|$ for all $e \in \mathbf{E}$.*

Proof. Real Case. First we show that $f \in \mathbf{F}^*$ can be extended with the given property to $\mathbf{F} \oplus \operatorname{span}\{e_0\}$, for a given $e_0 \notin \mathbf{F}$. For $e_1, e_2 \in \mathbf{F}$ we have

$$f(e_1) + f(e_2) = f(e_1 + e_2) \le \|e_1 + e_2\| \le \|e_1 + e_0\| + \|e_2 - e_0\|,$$

so that

$$f(e_2) - \|e_2 - e_0\| \le \|e_1 + e_0\| - f(e_1),$$

and hence

$$\sup\{\, f(e_2) - \|e_2 - e_0\| \mid e_2 \in \mathbf{F} \,\} \le \inf\{\, \|e_1 + e_0\| - f(e_1) \mid e_1 \in \mathbf{F} \,\}.$$

Let $a \in \mathbb{R}$ be any number between the sup and inf in the preceding expression and define $f' : \mathbf{F} \oplus \operatorname{span}\{e_0\} \to \mathbb{R}$ by $f'(e + te_0) = f(e) + ta$. It is clear that f' is linear and that $f'|\mathbf{F} = f$. To show that $|f'(e + te_0)| \le \|e + te_0\|$, note that by the definition of a,

$$f(e_2) - \|e_2 - e_0\| \le a \le \|e_1 + e_0\| - f(e_1),$$

so that by multiplying the second inequality by $t \ge 0$ and the first by $t < 0$, we get the desired result.

Second, one verifies that the set $\mathcal{S} = \{\, (\mathbf{G}, g) \mid \mathbf{F} \subset \mathbf{G} \subset \mathbf{E},\ \mathbf{G}$ is a subspace of $\mathbf{E},\ g \in \mathbf{G}^*,\ g|\mathbf{F} = f$, and $|g(e)| \le \|e\|$ for all $e \in \mathbf{G} \,\}$ is inductively ordered with respect to the ordering

$$(\mathbf{G}_1, g_1) \le (\mathbf{G}_2, g_2) \quad \textit{iff} \quad \mathbf{G}_1 \subset \mathbf{G}_2 \text{ and } g_2|\mathbf{G}_1 = g_1.$$

Thus by Zorn's lemma there exists a maximal element $(\mathbf{F}_0, f_0)$ of $\mathcal{S}$.

Third, using the first step and the maximality of $(\mathbf{F}_0, f_0)$, one concludes that $\mathbf{F}_0 = \mathbf{E}$.

Complex Case. Let $f = \operatorname{Re} f + i \operatorname{Im} f$ and note that complex linearity implies that $(\operatorname{Im} f)(e) = -(\operatorname{Re} f)(ie)$ for all $e \in \mathbf{F}$. By the real case, $\operatorname{Re} f$ extends to a real linear continuous map $(\operatorname{Re} f) : \mathbf{E} \to \mathbb{R}$, such that $|(\operatorname{Re} f)'(e)| \le \|e\|$ for all $e \in \mathbf{E}$. Define $f' : \mathbf{E} \to \mathbb{C}$ by $f'(e) = (\operatorname{Re} f)'(e) - i(\operatorname{Re} f)'(ie)$ and note that f is complex linear and $f'|\mathbf{F} = f$.

To show that $|f'(e)| \le \|e\|$ for all $e \in \mathbf{E}$, write $f'(e) = |f'(e)| \exp(i\theta)$, so complex linearity of f' implies $f'(e \cdot \exp(-i\theta)) \in \mathbb{R}$, and hence

$$|f'(e)| = f'(e \cdot \exp(-i\theta)) = (\operatorname{Re} f)'(e \cdot \exp(-i\theta)) \le \|e \cdot \exp(-i\theta)\| = \|e\|. \qquad \blacksquare$$

2.2.13 Corollary. *Let $(\mathbf{E}, \|\cdot\|)$ be a normed space, $\mathbf{F} \subset \mathbf{E}$ a subspace, and $f \in \mathbf{F}^*$ (the topological dual). Then there exists $f' \in \mathbf{E}^*$ such that $f'|\mathbf{F} = f$ and $\|f'\| = \|f\|$.*

Proof. We can assume $f \neq 0$. Then $|||e||| = \|f\|\,\|e\|$ is a norm on $\mathbf{E}$ and $|f(e)| \leq \|f\|\,\|e\| = |||e|||$ for all $e \in \mathbf{F}$. Applying the preceding theorem we get a linear map $f' : \mathbf{E} \to \mathbb{R}$ (or $\mathbb{C}$) with the properties $f'|\mathbf{F} = f$ and $|f'(e)| \leq |||e|||$ for all $e \in \mathbf{E}$. This says that $\|f'\| \leq \|f\|$, and since f' extends f, it follows that $\|f\| \leq \|f'\|$; that is, $\|f'\| = \|f\|$ and $f' \in \mathbf{E}^*$. ■

Applying the corollary to the linear function $ae \mapsto a$, for $e \in \mathbf{E}$ a fixed element, we get the following.

2.2.14 Corollary. *Let $\mathbf{E}$ be a normed vector space and $e \neq 0$. Then there exists $f \in \mathbf{E}^*$ such that $f(e) \neq 0$. In other words if $f(e) = 0$ for all $f \in \mathbf{E}^*$, then $e = 0$; that is, $\mathbf{E}^*$separates points of $\mathbf{E}$.*

Open Mapping Theorem. This result states that surjective linear maps are open.

2.2.15 Theorem (Open Mapping Theorem of Banach–Schauder)**.** *Let $\mathbf{E}$ and $\mathbf{F}$ be Banach spaces and suppose $A \in L(\mathbf{E}, \mathbf{F})$ is onto. Then A is an open mapping.*

Proof. To show A is an open mapping, it suffices to prove that the set $A(\mathrm{cl}(D_1(0)))$ contains a disk centered at zero in $\mathbf{F}$. Let $r > 0$. Since

$$\mathbf{E} = \bigcup_{n \geq 1} D_{nr}(0),$$

it follows that

$$\mathbf{F} = \bigcup_{n \geq 1} (A(D_{nr}(0)))$$

and hence

$$\bigcup_{n \geq 1} \mathrm{cl}(A(D_{nr}(0))) = \mathbf{F}.$$

Completeness of $\mathbf{F}$ implies that at least one of the sets $\mathrm{cl}(A(D_{nr}(0)))$ has a nonempty interior by the Baire category theorem 1.7.3. Because the mapping $e \in \mathbf{E} \mapsto ne \in \mathbf{E}$ is a homeomorphism, we conclude that $\mathrm{cl}(A(D_r(0)))$ contains some open set $V \subset \mathbf{F}$. We shall prove that the origin of $\mathbf{F}$ is in $\mathrm{int}\{\mathrm{cl}[A(D_r(0))]\}$ for some $r > 0$. Continuity of $(e_1, e_2) \in \mathbf{E} \times \mathbf{E} \mapsto e_1 - e_2 \in \mathbf{E}$ assures the existence of an open set $U \subset \mathbf{E}$ such that

$$U - U = \{\, e_1 - e_2 \mid e_1, e_2 \in U \,\} \subset D_r(0).$$

Choose $r > 0$ such that $D_r(0) \subset U$. Then

$$\mathrm{cl}(A(D_r(0))) \supset \mathrm{cl}(A(U) - A(U)) \supset \mathrm{cl}(A(U)) - \mathrm{cl}(A(U)) \supset V - V.$$

But

$$V - V = \bigcup_{e \in V} (V - e)$$

is open and clearly contains $0 \in \mathbf{F}$. It follows that there exists a disk $D_t(0) \subset \mathbf{F}$ such that $D_t(0) \subset \mathrm{cl}(A(D_r(0)))$.

Now let $\varepsilon(n) = 1/2^{n+1}$, $n = 0, 1, 2, \ldots$, so that $1 = \sum_{n \geq 0} \varepsilon(n)$. By the foregoing result for each n there exists an $\eta(n) > 0$ such that $D_{\eta(n)}(0) \subset \mathrm{cl}(A(D_{\varepsilon(n)}(0)))$. Clearly $\eta(n) \to 0$. We shall prove that $D_{\eta(0)} \subset A(\mathrm{cl}(D1(0)))$. For $v \in D_{\eta(0)}(0) \subset \mathrm{cl}(A(D_{\varepsilon(0)}(0)))$ there exists $e_0 \in D_{\varepsilon(0)}(0)$ such that $\|v - Ae_0\| < \eta(1)$ and thus $v - Ae_0 \in \mathrm{cl}(A(D_{\varepsilon(1)}(0)))$, so there exists $e_1 \in D_{\varepsilon(1)}(0)$ such that $\|v - Ae_0 - Ae_1\| < \eta(2)$, etc.

Inductively one constructs a sequence $e_n \in D_{\eta(n)}$ such that $\|v - Ae_0 - \cdots - Ae_n\| < \eta(n+1)$. The series $\sum_{n\geq 0} e_n$ is convergent because

$$\left\| \sum_{i=n+1}^{m} e_i \right\| \leq \sum_{i=n+1}^{m} \frac{1}{2^{i+1}}, \quad \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} = 1,$$

and $\mathbf{E}$ is complete. Let $e = \sum_{n\geq 0} e_n \in \mathbf{E}$. Thus,

$$Ae = \sum_{n=0}^{\infty} Ae_n = v,$$

and

$$\|e\| \leq \sum_{n=0}^{\infty} \|e_n\| \leq \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} = 1;$$

that is, $v \in D_{\eta(0)}(0)$ implies $v = Ae$, $\|e\| \leq 1$. Therefore,

$$D_{\eta(0)}(0) \subset A(\mathrm{cl}(D_1(0))). \qquad \blacksquare$$

An important consequence is the following.

2.2.16 Theorem (Banach's Isomorphism Theorem). *A continuous linear isomorphism of Banach spaces is a homeomorphism.*

Thus, if $\mathbf{F}$ and $\mathbf{G}$ are closed subspaces of the Banach space $\mathbf{E}$ and $\mathbf{E}$ is the *algebraic* direct sum of $\mathbf{F}$ and $\mathbf{G}$, then the mapping $(e, e') \in \mathbf{F} \times \mathbf{G} \mapsto e + e' \in \mathbf{E}$ is a continuous isomorphism, and hence a homeomorphism; that is, $\mathbf{E} = \mathbf{F} \oplus \mathbf{G}$; this proves the comment at the beginning of Supplement 2.1B.

Closed Graph Theorem. This result characterizes continuity by closedness of the graph of a linear map.

2.2.17 Theorem (Closed Graph Theorem). *Suppose that $\mathbf{E}$ and $\mathbf{F}$ are Banach spaces. A linear map $A : \mathbf{E} \to \mathbf{F}$ is continuous iff its graph*

$$\Gamma_A = \{ (e, Ae) \in \mathbf{E} \times \mathbf{F} \mid e \in \mathbf{E} \}$$

is a closed subspace of $\mathbf{E} \oplus \mathbf{F}$.

Proof. It is readily verified that Γ_A is a linear subspace of $\mathbf{E} \oplus \mathbf{F}$. If $A \in L(\mathbf{E}, \mathbf{F})$, then Γ_A is closed (see Exercise 1.4-2). Conversely, if Γ_A is closed, then it is a Banach subspace of $\mathbf{E} \oplus \mathbf{F}$, and since the mapping $(e, Ae) \in \Gamma_A \mapsto e \in \mathbf{E}$ is a continuous isomorphism, its inverse $e \in \mathbf{E} \mapsto (e, Ae) \in \Gamma_A$ is also continuous by Theorem 2.2.16. Since $(e, Ae) \in \Gamma_A \mapsto Ae \in \mathbf{F}$ is clearly continuous, so is the composition $e \mapsto (e, Ae) \mapsto Ae$. $\blacksquare$

The Closed graph theorem is often used in the following way. To show that a linear map $A : \mathbf{E} \to \mathbf{F}$ is continuous for $\mathbf{E}$ and $\mathbf{F}$ Banach spaces, it suffices to show that if $e_n \to 0$ and $Ae_n \to e'$, then $e' = 0$.

2.2.18 Corollary. *Let $\mathbf{E}$ be a Banach space and $\mathbf{F}$ a closed subspace of $\mathbf{E}$. Then $\mathbf{F}$ is split iff there exists $P \in L(\mathbf{E}, \mathbf{E})$ such that $P \circ P = P$ and $\mathbf{F} = \{ e \in \mathbf{E} \mid Pe = e \}$.*

Proof. If such a P exists, then clearly $\ker(P)$ is a closed subspace of $\mathbf{E}$ that is an algebraic complement of $\mathbf{F}$; any $e \in \mathbf{E}$ is of the form $e = e - Pe + Pe$ with $e - Pe \in \ker(P)$ and $Pe \in \mathbf{F}$.

Conversely, if $\mathbf{E} = \mathbf{F} \oplus \mathbf{G}$, define $P : \mathbf{E} \to \mathbf{E}$ by $P(e) = e_1$, where $e = e_1 + e_2$, $e_1 \in \mathbf{F}$, $e_2 \in \mathbf{G}$. P is clearly linear, $P^2 = P$, and $\mathbf{F} = \{ e \in \mathbf{E} \mid Pe = e \}$, so all there is to show is that P is continuous. Let $e_n = e_{1n} + e_{2n} \to 0$ and $P(e_n) = e_{1n} \to e'$; that is, $-e_{2n} \to e'$, and since $\mathbf{F}$ and $\mathbf{G}$ are closed this implies that $e' \in \mathbf{F} \cap \mathbf{G} = \{0\}$. By the closed graph theorem, $P \in L(\mathbf{E}, \mathbf{E})$. $\blacksquare$

2.2.19 Theorem (Fundamental Isomorphism Theorem)**.** *Let $A \in L(\mathbf{E}, \mathbf{F})$ be surjective where $\mathbf{E}$ and $\mathbf{F}$ are Banach spaces. Then $\mathbf{E}/\ker A$ and $\mathbf{F}$ are isomorphic Banach spaces.*

Proof. The map $[e] \mapsto Ae$ is bijective and continuous (since its norm is $\leq \|A\|$), so it is a homeomorphism. ■

A sequence of maps

$$\cdots \to \mathbf{E}_{i-1} \xrightarrow{A_i} \mathbf{E}_i \xrightarrow{A_{i+1}} \mathbf{E}_{i+1} \to \cdots$$

of Banach spaces is said to be ***split exact*** if for all i, $\ker A_{i+1} = \operatorname{range} A_i$ and both $\ker A_i$ and range A_i split. With this terminology, Theorem 2.2.19 can be reformulated in the following way: If $0 \to \mathbf{G} \to \mathbf{E} \to \mathbf{F} \to 0$ *is a split exact sequence of Banach spaces, then $\mathbf{E}/\mathbf{G}$ is a Banach space isomorphic to $\mathbf{F}$ (thus $\mathbf{F} \cong \mathbf{G} \oplus \mathbf{F}$).*

Uniform Boundedness Principle. Next we prove the ***uniform boundedness principle of Banach and Steinhaus***, the third pillar of linear analysis.

2.2.20 Theorem. *Let $\mathbf{E}$ and $\mathbf{F}$ be normed vector spaces, with $\mathbf{E}$ complete, and let $\{A_i\}_{i \in I} \subset L(\mathbf{E}, \mathbf{F})$. If for each $e \in \mathbf{E}$ the set $\{\|A_i e\|\}_{i \in I}$ is bounded in $\mathbf{F}$, then $\{\|A_i\|\}_{i \in I}$ is a bounded set of real numbers.*

Proof. Let $\varphi(e) = \sup\{\, \|A_i e\| \mid i \in I \,\}$ and note that

$$S_n = \{\, e \in \mathbf{E} \mid \varphi(e) \leq n \,\} = \bigcap_{i \in I} \{\, e \in \mathbf{E} \mid \|A_i e\| \leq n \,\}$$

is closed and $\bigcup_{n \geq 1} S_n = \mathbf{E}$. Since $\mathbf{E}$ is a complete metric space, the Baire category theorem 1.7.3 says that some S_n has nonempty interior; that is, there exist $r > 0$ and $e_0 \in \mathbf{E}$ such that $\varphi(e) \leq M$, for all $e \in \operatorname{cl}(D_r(e_0))$, where $M > 0$ is come constant.

For each $i \in I$ and $\|e\| = 1$, we have $\|A_i(re + e_0)\| \leq \varphi(re + e_0) \leq M$, so that

$$\begin{aligned}\|A_i e\| &= \frac{1}{r}\|A_i(re + e_0 - e_0)\| \leq \frac{1}{r}\|A_i(re + e_0)\| + \frac{1}{r}\|A_i e_0\| \\ &\leq \frac{1}{r}(M + \varphi(e_0)),\end{aligned}$$

that is, $\|A_i\| \leq (M + \varphi(e_0))/r$ for all $i \in I$. ■

2.2.21 Corollary. *If $\{A_n\} \subset L(\mathbf{E}, \mathbf{F})$ is a strongly convergent sequence (i.e., $lim_{n \to \infty} A_n e = Ae$ exists for every $e \in \mathbf{E}$), then $A \in L(\mathbf{E}, \mathbf{F})$.*

Proof. A is clearly a linear map. Since $\{A_n e\}$ is convergent, it is a bounded set for each $e \in \mathbf{E}$, so that by Theorem 2.2.20, $\{\|A_n\|\}$ is bounded by, say, $M > 0$. But then

$$\|Ae\| = \lim_{n \to \infty} \|A_n e\| \leq \lim_{n \to \infty} \sup \|A_n\| \, \|e\| \leq M\|e\|;$$

that is, $A \in L(\mathbf{E}, \mathbf{F})$. ■

Exercises

◇ **2.2-1.** If $\mathbf{E} = \mathbb{R}^n$ and $\mathbf{F} = \mathbb{R}^m$ with the standard norms, and $A : \mathbf{E} \to \mathbf{F}$ is a linear map, show that

(i) $\|A\|$ is the square root of the absolute value of the largest eigenvalue of AA^T, where A^T is the transpose of A, and

(ii) if $n, m \geq 2$, this norm does not come from an inner product.

HINT: Use Exercise 2.1-1.

◇ **2.2-2.** Let $\mathbf{E} = \mathbf{F} = \mathbb{R}^n$ with the standard norms and $A, B \in L(\mathbf{E}, \mathbf{F})$. Let $\langle A, B \rangle = \operatorname{trace}(AB^T)$. Show that this is an inner product on $L(\mathbf{E}, \mathbf{F})$.

◇ **2.2-3.** Show that the map

$$L(\mathbf{E}, \mathbf{F}) \times L(\mathbf{F}, \mathbf{E}) \to \mathbb{R}; \quad (A, B) \mapsto \operatorname{trace}(AB)$$

gives a (natural) isomorphism $L(\mathbf{E}, \mathbf{F})^* \cong L(\mathbf{F}, \mathbf{E})$.

◇ **2.2-4.** Let $\mathbf{E}, \mathbf{F}, \mathbf{G}$ be Banach spaces and $D \subset \mathbf{E}$ a linear subspace. A linear map $A : D \to \mathbf{F}$ is called ***closed*** if its graph Γ_A, the set of (x, Ax) where $x \in D$ is a closed subset of $\mathbf{E} \times \mathbf{F}$. If $A : D \subset \mathbf{E} \to \mathbf{F}$, and $B : D \subset \mathbf{E} \to \mathbf{G}$ are two closed operators with the same domain D, show that there are constants $M_1, M_2 > 0$ such that

$$\|Ae\| \leq M_1(\|Be\| + \|e\|) \quad \text{and} \quad \|Be\| \leq M_2(\|Ae\| + \|e\|)$$

for all $e \in \mathbf{E}$.
HINT: Norm $\mathbf{E} \oplus \mathbf{G}$ by $\|(e, g)\| = \|e\| + \|g\|$ and define $T : \Gamma_B \to \mathbf{G}$ by $T(e, Be) = Ae$. Use the closed graph theorem to show that $T \in L(\Gamma_B, \mathbf{G})$.

◇ **2.2-5** (Linear transversality). Let $\mathbf{E}, \mathbf{F}$ be Banach spaces, $\mathbf{F}_0 \subset \mathbf{F}$ a closed subspace, and $T \in L(\mathbf{E}, \mathbf{F})$. T is said to be ***transversal*** to $\mathbf{F}_0$, if $T^{-1}(\mathbf{F}_0)$ splits in $\mathbf{E}$ and $T(\mathbf{E}) + \mathbf{F}_0 = \{ Te + f \mid e \in \mathbf{E},\ f \in \mathbf{F} \} = \mathbf{F}$. Prove the following.

(i) T is transversal to $\mathbf{F}_0$ iff $\pi \circ T \in L(\mathbf{E}, \mathbf{F}/\mathbf{F}_0)$ is surjective with split kernel; here $\pi : \mathbf{F} \to \mathbf{F}/\mathbf{F}_0$ is the projection.

(ii) If $\pi \circ T \in L(\mathbf{E}, \mathbf{F}/\mathbf{F}_0)$ is surjective and $\mathbf{F}_0$ has finite codimension, then $\ker(\pi \circ T)$ has the same codimension and T is transversal to $\mathbf{F}_0$.

HINT: Use the algebraic isomorphism $T(\mathbf{E})/(\mathbf{F}_0 \cap T(\mathbf{E})) \cong (T(\mathbf{E}) + \mathbf{F}_0)/\mathbf{F}_0$ to show $\mathbf{E}/\ker(\pi \circ T) \cong \mathbf{F}/\mathbf{F}_0$; now use Corollary 2.2.18.

(iii) If $\pi \circ T \in L(\mathbf{E}, \mathbf{F}/\mathbf{F}_0)$ is surjective and if $\ker T$ and $\mathbf{F}_0$ are finite dimensional, then $\ker(\pi \circ T)$ is finite dimensional and T is transversal to $\mathbf{F}_0$.

HINT: Use the exact sequence $0 \to \ker T \to \ker(\pi \circ T) \to \mathbf{F}_0 \cap T(\mathbf{E}) \to 0$.

◇ **2.2-6.** Let $\mathbf{E}$ and $\mathbf{F}$ be Banach spaces. Prove the following.

(i) If $f \in \operatorname{cl}(\mathcal{S}([a, b], L(\mathbf{E}, \mathbf{F})))$ and $e \in \mathbf{E}$, then

$$\int_a^b f(t)e\, dt = \left(\int_a^b f(t)\, dt \right)(e)$$

HINT: $T \mapsto Te$ is in $L(L(\mathbf{E}, \mathbf{F}), \mathbf{F})$.

(ii) If $f \in \operatorname{cl}(\mathcal{S}([a, b], \mathbb{R})$ and $v \in \mathbf{F}$, then

$$\int_a^b f(t)v\, dt = \left(\int_a^b f(t)\, dt \right)(v)$$

HINT: $t \mapsto$ multiplication by t in $\mathbf{F}$ is in $L(\mathbb{R}, L(\mathbf{F}, \mathbf{F}))$; apply (i).

(iii) Let X be a topological space and $f : [a, b] \times X \to \mathbf{E}$ be continuous. Then the mapping

$$g : X \to \mathbf{E}, \quad g(x) = \int_a^b f(t, x)dt$$

is continuous.

HINT: For $t \in \mathbb{R}$, $x' \in X$ and $\varepsilon > 0$ given,

$$\|f(s, x) - f(t, x')\| < \varepsilon \quad \text{if } (s, x) \in U_1 \times U_{x',t};$$

use compactness of $[a, b]$ to find $U_{x'}$ as a finite intersection and such that $\|f(t, x) - f(t, x')\| < \varepsilon$ for all $t \in [a, b]$, $x \in U_{x'}$.

◇ **2.2-7.** Show that the Banach isomorphism theorem is false for normed incomplete vector spaces in the following way. Let $\mathbf{E}$ be the space of all polynomials over $\mathbb{R}$ normed as follows:

$$\|a_0 + a_1 x + \cdots + a_n x^n\| = \max\{|a_0|, \ldots, |a_n|\}.$$

(i) Show that $\mathbf{E}$ is not complete.

(ii) Define $A : \mathbf{E} \to \mathbf{E}$ by

$$A\left(\sum_{i=0}^n a_i x^i\right) = a_0 + \sum_{i=1}^n \frac{a_i}{i} x_i$$

and show that $A \in L(\mathbf{E}, \mathbf{E})$. Prove that $A^{-1} : \mathbf{E} \to \mathbf{E}$ exists.

(iii) Show that A^{-1} is not continuous.

◇ **2.2-8.** Let $\mathbf{E}$ and $\mathbf{F}$ be Banach spaces and $A \in L(\mathbf{E}, \mathbf{F})$. If $A(\mathbf{E})$ has finite codimension, show that it is closed.
HINT: If $\mathbf{F}_0$ is an algebraic complement to $A(\mathbf{E})$ in $\mathbf{F}$, show there is a continuous linear isomorphism $\mathbf{E}/\ker A \cong \mathbf{F}/\mathbf{F}_0$; compose its inverse with $\mathbf{E}/\ker A \to A(\mathbf{E})$.

◇ **2.2-9** (Symmetrization operator)**.** Define

$$\operatorname{Sym}^k : L^k(\mathbf{E}, \mathbf{F}) \to L^k(\mathbf{E}, \mathbf{F}),$$

by

$$\operatorname{Sym}^k A = \frac{1}{k!} \sum_{a \in S_k} \sigma A,$$

where $(\sigma A)(e_1, \ldots, e_k) = A(e_{\sigma(1)}, \ldots, e_{\sigma(k)})$. Show that:

(i) $\operatorname{Sym}^k(L^k(\mathbf{E}, \mathbf{F})) = L^k_s(\mathbf{E}, \mathbf{F})$.

(ii) $(\operatorname{Sym}^k)^2 = \operatorname{Sym}^k$.

(iii) $\|\operatorname{Sym}^k\| \le 1$.

(iv) If $\mathbf{F}$ is Banach, then $L^k_s(\mathbf{E}, \mathbf{F})$ splits in $L^k(\mathbf{E}, \mathbf{F})$.

HINT: Use Corollary 2.2.18.

(v) $(\operatorname{Sym}^k A)' = A'$.

◇ **2.2-10.** Show that a k-multilinear map continuous in each argument separately is continuous.
HINT: For $k = 2$: If $\|e_1\| \leq 1$, then $\|A(e_1, e_2)\| \leq \|A(\cdot, e_2)\|$, which by the uniform boundedness principle implies the inequality $\|A(e_1, \cdot)\| \leq M$ for $\|e_1\| \leq 1$.

◇ **2.2-11.** (i) Prove the ***Mazur-Ulam Theorem*** following the steps below (see Mazur and Ulam [1932], Banach [1932, p. 166]): *Every isometric surjective mapping* $\varphi : \mathbf{E} \to \mathbf{F}$ *such that* $\varphi(0) = 0$ *is a linear map.* Here $\mathbf{E}$ and $\mathbf{F}$ are normed vector spaces; φ being ***isometric*** means that $\|\varphi(x) - \varphi(y)\| = \|x - y\|$ for all $x, y \in \mathbf{E}$.

(a) Fix $x_1, x_2 \in \mathbf{E}$ and define

$$H_1 = \left\{ x \;\middle|\; \|x - x_1\| = \|x - x_2\| = \frac{1}{2}\,\|x_1 - x_2\| \right\},$$
$$H_n = \left\{ x \in H_{n-1} \;\middle|\; \|x - z\| \leq \frac{1}{2}\,\mathrm{diam}(H_{n-1}),\ z \in H_{n-1} \right\}.$$

Show that

$$\mathrm{diam}(H_n) \leq \frac{1}{2^{n-1}}\,\mathrm{diam}(H_1) \leq \frac{1}{2^{n-1}}\|x_1 - x_2\|.$$

Conclude that if $\bigcap_{n\geq 1} H_n \neq \varnothing$, then it consists of one point only.

(b) Show by induction that if $x \in H_n$, then $x_1 + x_2 - x \in H_n$.

(c) Show that $(x_1 + x_2)/2 = \bigcap_{n\geq 1} H_n$.
HINT: Show inductively that $(x_1 + x_2)/2 \in H_n$ using (b).

(d) From (c) deduce that

$$\varphi\left(\frac{1}{2}(x_1 + x_2)\right) = \frac{1}{2}(\varphi(x_1) + \varphi(x_2)).$$

Use $\varphi(0) = 0$ to conclude that φ is linear.

(ii) (Chernoff, 1970). The goal of this exercise is to study the Mazur–Ulam theorem, dropping the assumption that φ is onto, and replacing it with the assumption that φ is homogeneous: $\varphi(tx) = t\varphi(x)$ for all $t \in \mathbb{R}$ and $x \in \mathbf{E}$.

(a) A normed vector space is called ***strictly convex*** if equality holds in the triangle inequality only for colinear points. Show that if F is strictly convex, then φ is linear.
HINT:

$$\|\varphi(x) - \varphi(y)\| = \left\|\varphi(x) - \varphi\left(\frac{x+y}{2}\right)\right\| + \left\|\varphi(y) - \varphi\left(\frac{x+y}{2}\right)\right\|$$

and

$$\left\|\varphi(x) - \varphi\left(\frac{x+y}{2}\right)\right\| = \left\|\varphi(y) - \varphi\left(\frac{x+y}{2}\right)\right\|.$$

Show that

$$\varphi\left(\frac{x+y}{2}\right) = \frac{1}{2}(\varphi(x) + \varphi(y)).$$

(b) Show that, in general, the assumption on φ being onto is necessary by considering the following counterexample. Let $\mathbf{E} = \mathbb{R}^2$ and $\mathbf{F} = \mathbb{R}^3$, both with the max norm. Define $\varphi : \mathbf{E} \to \mathbf{F}$ by

$$\begin{aligned} \varphi(a,b) &= (a,b,\sqrt{ab}), && a,b>0;\\ \varphi(-a,b) &= (-a,b,-\sqrt{ab}), && a,b>0;\\ \varphi(a,-b) &= (a,-b,-\sqrt{ab}), && a,b>0;\\ \varphi(-a,b) &= (-a,-b,-\sqrt{ab}), && a,b>0. \end{aligned}$$

Show that φ is not linear, φ is homogeneous, φ is an isometry, and $\varphi(0,0) = (0,0,0)$.
Hint: Prove the inequality

$$|\alpha\beta - \gamma\delta| \leq \max(|\alpha^2 - \gamma^2|, |\beta^2 - \delta^2|).$$

◇ **2.2-12.** Let $\mathbf{E}$ be a complex n-dimensional vector space.

(i) Show that the set of all operators $A \in L(\mathbf{E},\mathbf{E})$ which have n distinct eigenvalues is open and dense in $\mathbf{E}$.

Hint: Let p be the characteristic polynomial of A, that is, $p(\lambda) = \det(A - \lambda I)$, and let $\mu_1,\ldots,\mu_{n-1}$ be the roots of p'. Then A has multiple eigenvalues iff $p(\mu_1)\cdots p(\mu_{n-1}) = 0$. The last expression is a symmetric polynomial in $\mu_1,\ldots,\mu_{n-1}$, and so is a polynomial in the coefficients of p' and therefore is a polynomial q in the entries of the matrix of A in a basis. Show that $q^{-1}(0)$ is the set of complex $n\times n$ matrices which have multiple eigenvalues; $q^{-1}(0)$ has open dense complement by Exercise 1.1-12.

(ii) Prove the ***Cayley–Hamilton Theorem***: *If p is the characteristic polynomial of $A \in L(\mathbf{E},\mathbf{E})$, then $p(A) = 0$.*

Hint: If the eigenvalues of A are distinct, show that the matrix of A in the basis of eigenvectors $e_1,\ldots,e_n$ is diagonal. Apply A, A^2, ..., A^{n-1}. Then show that for any polynomial q the matrix of $q(A)$ in the same basis is diagonal with entries $q(\lambda_i)$, where λ_i are the eigenvalues of A. Finally, let $q = p$. If A is general, apply (i).

◇ **2.2-13.** Let $\mathbf{E}$ be a normed real (resp. complex) vector space.

(i) Show that $\lambda : \mathbf{E} \to \mathbb{R}$ (resp., $\mathbb{C}$) is continuous if and only if $\ker\lambda$ is closed.

Hint: Let $e \in \mathbf{E}$ satisfy $\lambda(e) = 1$ and choose a disk D of radius r centered at e such that $D \cap (e + \ker\lambda) = \varnothing$. Then $\lambda(x) \neq 1$ for all $x \in D$. Show that if $x \in D$ then $\lambda(x) < 1$. If not, let $\alpha = \lambda(x)$, $|\alpha| > 1$. Then $\|x/\alpha\| < r$ and $\lambda(x/\alpha) = 1$.

(ii) Show that if $\mathbf{F}$ is a closed subspace of $\mathbf{E}$ and $\mathbf{G}$ is a finite dimensional subspace, then $\mathbf{G}+\mathbf{F}$ is closed.

Hint: Assume $\mathbf{G}$ is one dimensional and generated by g. Write any $x \in \mathbf{G}+\mathbf{F}$ as $x = \lambda(x)g + f$ and use (i) to show λ is continuous on $\mathbf{G}+\mathbf{F}$.

◇ **2.2-14.** Let $\mathbf{F}$ be a Banach space.

(i) Show that if $\mathbf{E}$ is a finite dimensional subspace of $\mathbf{F}$, then $\mathbf{E}$ is split.

Hint: Define $P : \mathbf{F} \to \mathbf{F}$ by

$$P(x) = \sum_{i=1,\ldots,n} e^i(x)e_i,$$

where $\{e_1,\ldots,e_n\}$ is a basis of $\mathbf{E}$ and $\{e^1,\ldots,e^n\}$ is a dual basis, that is, $e^i(e_j) = \delta_{ij}$. Then use Corollary 2.2.18.

(ii) Show that if $\mathbf{E}$ is closed and finite codimensional, then it is split.

(iii) Show that if $\mathbf{E}$ is closed and contains a finite-codimensional subspace $\mathbf{G}$ of $\mathbf{F}$, then it is split.

(iv) Let $\lambda : \mathbf{F} \to \mathbb{R}$ be a linear discontinuous map and let $\mathbf{E} = \ker \lambda$. Show that the codimension of $\mathbf{E}$ is 1 and that $\mathbf{E}$ is not closed. Thus finite codimensional subspaces of $\mathbf{F}$ are not necessarily closed. Compare this with (i) and (ii), and with Exercise 2.2-8.

◇ **2.2-15.** Let $\mathbf{E}$ and $\mathbf{F}$ be Banach spaces and $T \in L(\mathbf{E}, \mathbf{F})$. Define $T^* : \mathbf{F}^* \to \mathbf{E}^*$ by $\langle T^*\beta, e\rangle = \langle \beta, Te\rangle$ for $e \in \mathbf{E}$, $\beta \in \mathbf{F}^*$. Show that:

(i) $T^* \in L(\mathbf{F}^*, \mathbf{E}^*)$ and $T^{**}|\mathbf{E} = T$.

(ii) $\ker T^* = T(\mathbf{E})^{\mathrm{o}} := \{\, \beta \in \mathbf{F}^* \mid \langle \beta, Te\rangle = 0 \text{ for all } e \in \mathbf{E} \,\}$ and $\ker T = (T^*(\mathbf{F}^*))^{\mathrm{o}} := \{\, e \in \mathbf{E} \mid \langle T^*\beta, e\rangle = 0 \text{ for all } \beta \in \mathbf{F}^* \,\}$.

(iii) If $T(\mathbf{E})$ is closed, then $T^*(\mathbf{F}^*) = (\ker T)^{\mathrm{o}}$.

Hint: The induced map $\mathbf{E}/\ker T \to T(\mathbf{E})$ is a Banach space isomorphism; let S be its inverse. If $\lambda \in (\ker T)^{\mathrm{o}}$, define the element $\mu \in (\mathbf{E}/\ker T)^*$ by $\mu([e]) = \lambda(e)$. Let $\nu \in \mathbf{F}^*$ denote the extension of $S^*(\mu) \in (T(\mathbf{E}))^*$ to $\nu \in \mathbf{F}^*$ with the same norm and show that $T^*(\nu) = \lambda$.

(iv) If $T(\mathbf{E})$ is closed, then $\ker T^*$ is isomorphic to $(\mathbf{F}/T(\mathbf{E}))^*$ and $(\ker T)^*$ is isomorphic to $\mathbf{E}^*/T^*(\mathbf{F}^*)$.

2.3 The Derivative

Definition of the Derivative. For a differentiable function $f : U \subset \mathbb{R} \to \mathbb{R}$, the usual interpretation of the derivative at a point $u_0 \in U$ is the slope of the line tangent to the graph of f at u_0. To generalize this, we interpret $\mathbf{D}f(u_0) = f'(u_0)$ as a linear map acting on the vector $(u - u_0)$.

2.3.1 Definition. *Let $\mathbf{E}, \mathbf{F}$ be normed vector spaces, U be an open subset of $\mathbf{E}$ and let $f : U \subset \mathbf{E} \to \mathbf{F}$ a given maping. Let $u_0 \in U$. We say that f is **differentiable at** the point u_0 provided there is a bounded linear map $\mathbf{D}f(u_0) : \mathbf{E} \to \mathbf{F}$ such that for every $\epsilon > 0$, there is an $\delta > 0$ such that whenever $0 < \|u - u_0\| < \delta$, we have*

$$\frac{\|f(u) - f(u_0) - \mathbf{D}f(u_0)\cdot(u - u_0)\|}{\|u - u_0\|} < \epsilon,$$

where $\|\cdot\|$ represents the norm on the appropriate space and where the evaluation of $\mathbf{D}f(u_0)$ on $e \in \mathbf{E}$ is denoted $\mathbf{D}f(u_0)\cdot e$.

This definition can also be written as

$$\lim_{u \to u_0} \frac{f(u) - f(u_0) - \mathbf{D}f(u_0)\cdot(u - u_0)}{\|u - u_0\|} = 0.$$

We shall shortly show that the derivative is unique if it exists and embark on relating this notion to ones that are perhaps more familiar to the reader in Euclidean space; we shall also develop many familiar properties of the derivative. However, it is useful to rephrase the definition slightly first. We shall do this in terms of the notion of tangency.

Tangency of Maps. An alternative way to think of the derivative in one variable calculus it to say that $\mathbf{D}f(u_0)$ is the unique linear map from $\mathbb{R}$ into $\mathbb{R}$ such that the mapping $g : U \to \mathbb{R}$ given by

$$u \mapsto g(u) = f(u_0) + \mathbf{D}f(u_0)\cdot(u - u_0)$$

is tangent to f at u_0, as in Figure 2.3.1.

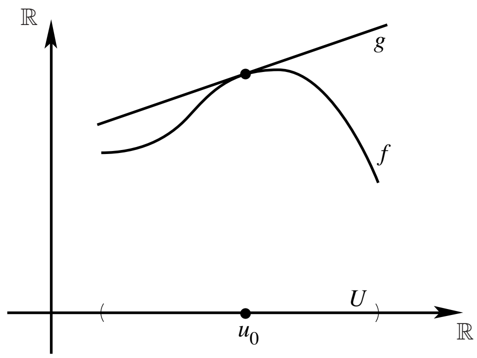

FIGURE 2.3.1. Derivative of a function of one variable

2.3.2 Definition. *Let* $\mathbf{E}, \mathbf{F}$ *be normed vector spaces, with maps* $f, g : U \subset \mathbf{E} \to \mathbf{F}$ *where* U *is open in* $\mathbf{E}$. *We say* f *and* g *are* ***tangent*** *at the point* $u_0 \in U$ *if* $f(u_0) = g(u_0)$ *and*

$$\lim_{u \to u_0} \frac{\|f(u) - g(u)\|}{\|u - u_0\|} = 0,$$

where $\|\cdot\|$ *represents the norm on the appropriate space.*

2.3.3 Proposition. *For* $f : U \subset \mathbf{E} \to \mathbf{F}$ *and* $u_0 \in U$ *there is at most one* $L \in L(\mathbf{E}, \mathbf{F})$ *such that the map* $g_L : U \subset \mathbf{E} \to \mathbf{F}$ *given by* $g_L(u) = f(u_0) + L(u - u_0)$ *is tangent to* f *at* u_0.

Proof. Let L_1 and $L_2 \in L(\mathbf{E}, \mathbf{F})$ satisfy the conditions of the proposition. If $e \in \mathbf{E}$, $\|e\| = 1$, and $u = u_0 + \lambda e$ for $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), then for $\lambda \neq 0$, small $u \in U$, and we have

$$\begin{aligned} \|L_1 e - L_2 e\| &= \frac{\|L_1(u - u_0) - L_2(u - u_0)\|}{\|u - u_0\|} \\ &\leq \frac{\|f(u) - f(u_0) - L_1(u - u_0)\|}{\|u - u_0\|} \\ &\quad + \frac{\|f(u) - f(u_0) - L_2(u - u_0)\|}{\|u - u_0\|}. \end{aligned}$$

As $\lambda \to 0$, the right hand side approaches zero so that $\|(L_1 - L_2)e\| = 0$ for all $e \in \mathbf{E}$ satisfying $\|e\| = 1$; therefore, $\|L_1 - L_2\| = 0$ and thus $L_1 = L_2$. ■

We can thus rephrase the definition of the derivative this way: If, in Proposition 2.3.3, there is such an $L \in L(\mathbf{E}, \mathbf{F})$, then f is differentiable at u_0, and the derivative of f at u_0 is $\mathbf{D}f(u_0) = L$. Thus, the derivative, if it exists, is unique.

2.3.4 Definition. *If* f *is differentiable at each* $u_0 \in U$, *the map*

$$\mathbf{D}f : U \to L(\mathbf{E}, \mathbf{F}); \quad u \mapsto \mathbf{D}f(u)$$

is called the ***derivative*** *of* f. *Moreover, if* $\mathbf{D}f$ *is a continuous map (where* $L(\mathbf{E}, \mathbf{F})$ *has the norm topology), we say* f *is of class* C^1 *(or is* ***continuously differentiable****). Proceeding inductively we define*

$$\mathbf{D}^r f := \mathbf{D}(\mathbf{D}^{r-1} f) : U \subset \mathbf{E} \to L^r(\mathbf{E}, \mathbf{F})$$

if it exists, where we have identified $L(\mathbf{E}, L^{r-1}(\mathbf{E}, \mathbf{F}))$ *with* $L^r(\mathbf{E}, \mathbf{F})$ *(see Proposition* 2.2.9*). If* $\mathbf{D}^r f$ *exists and is norm continuous, we say* f *is of* ***class*** C^r.

Basic Properties of the Derivative. We shall reformulate the definition of the derivative with the aid of the somewhat imprecise but very convenient ***Landau symbol*** : $o(e^k)$ *will denote a continuous function of* e *defined in a neighborhood of the origin of a normed vector space* $\mathbf{E}$*, satisfying* $\lim_{e\to 0}(o(e^k)/\|e\|^k) = 0$. The collection of these functions forms a vector space. Clearly $f : U \subset \mathbf{E} \to \mathbf{F}$ is differentiable at $u_0 \in U$ iff there exists a linear map $\mathbf{D}f(u_0) \in L(\mathbf{E}, \mathbf{F})$ such that

$$f(u_0 + e) = f(u_0) + \mathbf{D}f(u_0) \cdot e + o(e).$$

Let us use this notation to show that if $\mathbf{D}f(u_0)$ *exists, then* f *is continuous at* u_0:

$$\lim_{e\to 0} f(u_0 + e) = \lim_{e\to 0}(f(u_0) + \mathbf{D}f(u_0) \cdot e + o(e)) = f(u_0).$$

2.3.5 Proposition (Linearity of the Derivative)**.** *Let* $f, g : U \subset \mathbf{E} \to \mathbf{F}$ *be* r *times differentiable mappings and* a *a real* (*or complex*) *constant. Then* af *and* $f + g : U \subset \mathbf{E} \to \mathbf{F}$ *are* r *times differentiable with*

$$\mathbf{D}^r(f + g) = \mathbf{D}^r f + \mathbf{D}^r g \quad \textit{and} \quad \mathbf{D}^r(af) = a\mathbf{D}^r f.$$

Proof. If $u \in U$ and $e \in \mathbf{E}$, then

$$\begin{aligned} f(u + e) &= f(u) + \mathbf{D}f(u) \cdot e + o(e) \text{ and} \\ g(u + e) &= g(u) + \mathbf{D}g(u) \cdot e + o(e), \end{aligned}$$

so that adding these two relations yields

$$(f + g)(u + e) = (f + g)(u) + (\mathbf{D}f(u) + \mathbf{D}g(u)) \cdot e + o(e).$$

The case $r > 1$ follows by induction. Similarly,

$$af(u + e) = af(u) + a\mathbf{D}f(u) \cdot e + ao(e) = af(u) + a\mathbf{D}f(u) \cdot e + o(e).$$

■

2.3.6 Proposition (Derivative of a Cartesian Product)**.** *Let* $f_i : U \subset \mathbf{E} \to \mathbf{F}_i$*,* $1 \le i \le n$*, be a collection of* r *times differentiable mappings. Then* $f = f_1 \times \cdots \times f_n : U \subset \mathbf{E} \to \mathbf{F}_1 \times \cdots \times \mathbf{F}_n$ *defined by* $f(u) = (f_1(u), \ldots, f_n(u))$ *is* r *times differentiable and*

$$\mathbf{D}^r f = \mathbf{D}^r f_1 \times \cdots \times D^r f_n.$$

Proof. For $u \in U$ and $e \in \mathbf{E}$, we have

$$\begin{aligned} f(u + e) &= (f_1(u + e), \ldots, f_n(u + e)) \\ &= (f_1(u) + \mathbf{D}f_1(u) \cdot e + o(e), \ldots, f_n(u) + \mathbf{D}f_n(u) \cdot e + o(e)) \\ &= (f_1(u), \ldots, f_n(u)) + (\mathbf{D}f_1(u), \ldots, \mathbf{D}f_n(u)) \cdot e \\ &\quad + (o(e), \ldots, o(e)) \\ &= f(u) + \mathbf{D}f(u) \cdot e + o(e), \end{aligned}$$

the last equality follows using the sum norm in $\mathbf{F}_1 \times \cdots \times \mathbf{F}_n$:

$$\|(o(e), \ldots, o(e))\| = \|o(e)\| + \cdots + \|o(e)\|,$$

so $(o(e), \ldots, o(e)) = o(e)$. ■

Notice from the definition that for $L \in L(\mathbf{E}, \mathbf{F})$, $\mathbf{D}L(u) = L$ for any $u \in \mathbf{E}$. It is also clear that the derivative of a constant map is zero.

Usually all our spaces will be real and linearity will mean real-linearity. In the complex case, differentiable mappings are the subject of analytic function theory, a subject we shall not pursue in this book (see Exercise 2.3-6 for a hint of why there is a relationship with analytic function theory).

Jacobian Matrices. In addition to the foregoing approach, there is a more traditional way to differentiate a function $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$. We write out f in component form using the following notation:

$$f(x^1, \ldots, x^n) = (f^1(x^1, \ldots, x^n), \ldots, f^m(x^1, \ldots, x^n))$$

and compute partial derivatives, $\partial f^j / \partial x^i$ for $j = 1, \ldots, m$ and $i = 1, \ldots, n$, where the symbol $\partial f^j / \partial x^i$ means that we compute the usual derivative of f^j with respect to x^i while keeping the other variables

$$x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n$$

fixed.

For $f : \mathbb{R} \to \mathbb{R}$, $\mathbf{D}f(x)$ is just the linear map "multiplication by df/dx," that is, $df/dx = \mathbf{D}f(x) \cdot 1$. This fact, which is obvious from the definitions, can be generalized to the following theorem.

2.3.7 Proposition. *Suppose that $U \subset \mathbb{R}^n$ is an open set and that $f : U \to \mathbb{R}^m$ is differentiable. Then the partial derivatives $\partial f^j / \partial x^i$ exist, and the matrix of the linear map $\mathbf{D}f(x)$ with respect to the standard bases in $\mathbb{R}^n$ and $\mathbb{R}^m$ is given by*

$$\begin{bmatrix} \dfrac{\partial f^1}{\partial x^1} & \dfrac{\partial f^1}{\partial x^2} & \cdots & \dfrac{\partial f^1}{\partial x^n} \\ \dfrac{\partial f^2}{\partial x^1} & \dfrac{\partial f^2}{\partial x^2} & \cdots & \dfrac{\partial f^2}{\partial x^n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial f^m}{\partial x^1} & \dfrac{\partial f^m}{\partial x^2} & \cdots & \dfrac{\partial f^m}{\partial x^n} \end{bmatrix},$$

where each partial derivative is evaluated at $x = (x^1, \ldots, x^n)$. This matrix is called the ***Jacobian matrix*** *of f.*

Proof. By the usual definition of the matrix of a linear mapping from linear algebra, the (j, i)th matrix element a_i^j of $\mathbf{D}f(x)$ is given by the jth component of the vector $\mathbf{D}f(x) \cdot e_i$, where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. Letting $y = x + he_i$, we see that

$$\begin{aligned} &\frac{\|f(y) - f(x) - \mathbf{D}f(x)(y - x)\|}{\|y - x\|} \\ &= \frac{1}{|h|} \left\| f(x^1, \ldots, x^i + h, \ldots, x^n) - f(x^1, \ldots, x^n) - h\mathbf{D}f(x)e_i \right\| \end{aligned}$$

approaches zero as $h \to 0$, so the jth component of the numerator does as well; that is,

$$\lim_{h \to 0} \frac{1}{|h|} \left| f^j(x^1, \ldots, x^i + h, \ldots, x^n) - f^j(x^1, \ldots, x^n) - ha_i^j \right| = 0,$$

which means that $a_i^j = \partial f^j / \partial x^i$. ■

In computations one can usually compute the Jacobian matrix easily, and this proposition then gives $\mathbf{D}f$. In some books, $\mathbf{D}f$ is called the ***differential*** or the ***total derivative*** of f.

2.3.8 Example. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x, y) = (x^2, x^3 y, x^4 y^2)$. Then $\mathbf{D}f(x, y)$ is the linear map whose matrix in the standard basis is

$$\begin{bmatrix} \dfrac{\partial f^1}{\partial x} & \dfrac{\partial f^1}{\partial y} \\ \dfrac{\partial f^2}{\partial x} & \dfrac{\partial f^2}{\partial y} \\ \dfrac{\partial f^3}{\partial x} & \dfrac{\partial f^3}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 0 \\ 3x^2 y & x^3 \\ 4x^3 y^2 & 2x^4 y \end{bmatrix},$$

where $f^1(x,y) = x^2$, $f^2(x,y) = x^3y$, $f^3(x,y) = x^4y^2$. ♦

One should take special note when $m = 1$, in which case we have a real–valued function of n variables. Then $\mathbf{D}f$ has the matrix

$$\left[\frac{\partial f}{\partial x^1} \cdots \frac{\partial f}{\partial x^n}\right]$$

and the derivative applied to a vector $e = (a^1, \ldots, a^n)$ is

$$\mathbf{D}f(x) \cdot e = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} a^i.$$

The Gradient and Differential. It should be emphasized that $\mathbf{D}f$ assigns a linear mapping to each $x \in U$ and the definition of $\mathbf{D}f(x)$ is independent of the basis used. If we change the basis from the standard basis to another one, the matrix elements will of course change. If one examines the definition of the matrix of a linear transformation, it can be seen that the columns of the matrix relative to the new basis will be the derivative $\mathbf{D}f(x)$ applied to the new basis in $\mathbb{R}^n$ with this image vector expressed in the new basis in $\mathbb{R}^m$. Of course, the linear map $\mathbf{D}f(x)$ itself does not change from basis to basis. In the case $m = 1$, $\mathbf{D}f(x)$ is, in the standard basis, a $1 \times n$ matrix. The vector whose components are the same as those of $\mathbf{D}f(x)$ is called the ***gradient*** of f, and is denoted grad f or ∇f. Thus for $f : U \subset \mathbb{R}^n \to \mathbb{R}$,

$$\operatorname{grad} f = \left[\frac{\partial f}{\partial x^1}, \cdots, \frac{\partial f}{\partial x^n}\right]$$

(Sometimes it is said that grad f is just $\mathbf{D}f$ with commas inserted!) The formation of gradients makes sense in a general inner product space as follows.

2.3.9 Definition. (i) *Let $\mathbf{E}$ be a normed space and $f : U \subset \mathbf{E} \to \mathbb{R}$ be differentiable so that $\mathbf{D}f(u) \in L(\mathbf{E}, \mathbb{R}) = \mathbf{E}^*$. In this case we sometimes write $\mathbf{d}f(u)$ for $\mathbf{D}f(u)$ and call $\mathbf{d}f$ the **differential** of f. Thus $\mathbf{d}f : U \to \mathbf{E}^*$.*

(ii) *If $\mathbf{E}$ is a Hilbert space, the **gradient** of f is the map*

$$\operatorname{grad} f = \nabla f : U \to \mathbf{E} \quad \textit{defined by } \langle \nabla f(u), e\rangle = \mathbf{d}f(u) \cdot e,$$

where $\mathbf{d}f(u) \cdot e$ means the linear map $\mathbf{d}f(u)$ applied to the vector e.

Note that the existence of $\nabla f(u)$ requires the Riesz representation theorem (see Theorem 2.2.5). The notation $\delta f/\delta u$ instead of $(\operatorname{grad} f)(u) = \nabla f(u)$ is also in wide use, especially in the case in which $\mathbf{E}$ is a space of functions. See Supplement 2.4C below.

2.3.10 Example. Let $(\mathbf{E}, \langle , \rangle)$ be a real inner product space and let $f(u) = \|u\|^2$. Since $\|u\|^2 = \|u_0\|^2 + 2\langle u_0, u - u_0\rangle + \|u - u_0\|^2$, we obtain $\mathbf{d}f(u_0) \cdot e = 2\langle u_0, e\rangle$ and thus $\nabla f(u) = 2u$. Hence f is of class C^1. But since $\mathbf{D}f(u) = 2\langle u, \cdot\rangle \in \mathbf{E}^*$ is a continuous linear map in $u \in \mathbf{E}$, it follows that $\mathbf{D}^2 f(u) = \mathbf{D}f \in L(\mathbf{E}, \mathbf{E}^*)$ and thus $\mathbf{D}^k f = 0$ for $k \geq 3$. Thus f is of class C^∞. The mapping f considered here is a special case of a polynomial mapping (see Definition 2.2.10). ♦

Fundamental Theorem. We close this section with the fundamental theorem of calculus in real Banach spaces. First a bit of notation. If $\varphi : U \subset \mathbb{R} \to \mathbf{F}$ is differentiable, then $\mathbf{D}\varphi(t) \in L(\mathbb{R}, \mathbf{F})$. The space $L(\mathbb{R}, \mathbf{F})$ is isomorphic to $\mathbf{F}$ by $A \mapsto A(1)$, $1 \in R$; note that $\|A\| = \|A(1)\|$. We denote

$$\varphi' = \frac{d\varphi}{dt} = \mathbf{D}\varphi(t) \cdot 1, \quad 1 \in \mathbb{R}$$
$$\varphi'(t) = \lim_{h \to 0} \frac{\varphi(t+h) - \varphi(t)}{h}$$

and φ is differentiable iff φ' exists.

2.3.11 Theorem (Fundamental Theorem of Calculus)**.**

(i) *If $g : [a,b] \to \mathbf{F}$ is continuous, where $\mathbf{F}$ is a real normed space, then the map*

$$f :]a,b[\to \mathbf{F} \quad \textit{defined by } f(t) = \int_a^t g(s)\, ds$$

is differentiable and we have $f' = g$.

(ii) *If $f : [a,b] \to \mathbf{F}$ is continuous, is differentiable on $]a,b[$ and f' extends to a continuous map on $[a,b]$, then*

$$f(b) - f(a) = \int_a^b f'(s)\, ds.$$

Proof. (i) Let $t_0 \in]a,b[$. Since the integral is linear and continuous,

$$\|f(t_0+h) - f(t_0) - hg(t_0)\| = \left\| \int_{t_0}^{t_0+h} (g(s) - g(t_0)) ds \right\| \le |h| L_{g,h},$$

where $L_{g,h} = \sup\{\, \|g(s) - g(t_0)\| \mid t_0 \le s \le t_0 + h \,\}$. However, $L_{g,h} \to 0$ as $|h| \to 0$ by continuity of g at t_0.

(ii) Let the function $h(t)$ be defined by

$$h(t) = \left(\int_a^t f'(s)\, ds \right) - f(t),$$

By (i), $h'(t) = 0$ on $]a,b[$ and h is continuous on $[a,b]$. If for some $t \in [a,b]$, $h(t) \neq h(a)$, then by the Hahn–Banach theorem there exists $\alpha \in \mathbf{F}^*$ such that $(\alpha \circ h)(t) \neq (\alpha \circ h)(a)$. Moreover, $\alpha \circ h$ is differentiable on $]a,b[$ and its derivative is zero (Exercise 2.3-4). Thus by elementary calculus, $\alpha \circ h$ is constant on $[a,b]$, a contradiction. Hence $h(t) = h(a)$ for all $t \in [a,b]$. In particular, $h(a) = h(b)$. ■

Exercises

◇ **2.3-1.** Let $B : \mathbf{E} \times \mathbf{F} \to \mathbf{G}$ be a continuous bilinear map of normed spaces. Show that B is C^∞ and that

$$\mathbf{D}B(u,v)(e,f) = B(u,f) + B(e,v).$$

◇ **2.3-2.** Show that the derivative of a map is unaltered if the spaces are renormed with equivalent norms.

◇ **2.3-3.** If $f \in S^k(\mathbf{E}, \mathbf{F})$, show that for, $i = 1, \ldots, k$,

$$\mathbf{D}^k f(0)(e_1, \ldots, e_k) = \left. \frac{\partial^k}{\partial t_1 \cdots \partial t_k} f(t_1 e_1 + \cdots + t_k e_k) \right|_{t_i = 0}$$

and

$$\mathbf{D}^i f(0) = 0 \quad \text{for } i = 1, \ldots, k-1.$$

◇ **2.3-4.** Let $f : U \subset \mathbf{E} \to \mathbf{F}$ be a differentiable (resp., C^r) map and $A \in L(\mathbf{F}, \mathbf{G})$. Show that $A \circ f : U \subset \mathbf{E} \to \mathbf{G}$ is differentiable (resp., C^r) and $\mathbf{D}^r(A \circ f)(u) = A \circ \mathbf{D}^r f(u)$.
Hint: Use induction.

◇ **2.3-5.** Let $f : U \subset \mathbf{E} \to \mathbf{F}$ be r times differentiable and $A \in L(\mathbf{G}, \mathbf{E})$. Show that

$$\mathbf{D}^i(f \circ A)(v) \cdot (g_1, \ldots, g_i) = \mathbf{D}^i f(Av) \cdot (Ag_1, \ldots, Ag_i)$$

exists for all $i \leq r$, where $v \in A^{-1}(U)$, and $g_1, \ldots, g_i \in \mathbf{G}$. Generalize to the case where A is an affine map.

◇ **2.3-6.** (i) Show that a *complex* linear map $A \in L(\mathbb{C}, \mathbb{C})$ is necessarily of the form $A(z) = \lambda z$, for some $\lambda \in \mathbb{C}$.

(ii) Show that the matrix of $A \in L(\mathbb{C}, \mathbb{C})$, when A is regarded as a *real* linear map in $L(\mathbb{R}^2, \mathbb{R}^2)$, is of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

HINT: $\lambda = a + ib$.

(iii) Show that a map $f : U \subset \mathbb{C} \to \mathbb{C}$, $f = g + ih$, $g, h : U \subset \mathbb{R}^2 \to \mathbb{R}$ is *complex* differentiable iff the ***Cauchy–Riemann equations***

$$\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y} \quad , \quad \frac{\partial g}{\partial y} = -\frac{\partial h}{\partial x}$$

are satisfied.

HINT: Use (ii) and Proposition 2.3.7.

◇ **2.3-7.** Let $(\mathbf{E}, \langle , \rangle)$ be a *complex* inner product space. Show that the map $f(u) = \|u\|^2$ is *not* differentiable. Contrast this with Example 2.3.10.
HINT: $\mathbf{D}f(u)$, if it exists, should equal $2\,\mathrm{Re}(\langle u, \cdot \rangle)$.

◇ **2.3-8.** Show that the matrix of $\mathbf{D}^2 f(x) \in L^2(\mathbb{R}^n, \mathbb{R})$ for $f : U \subset \mathbb{R}^n \to \mathbb{R}$, is given by

$$\begin{bmatrix} \dfrac{\partial^2 f}{\partial x^1 \partial x^1} & \dfrac{\partial^2 f}{\partial x^1 \partial x^2} & \cdots & \dfrac{\partial^2 f}{\partial x^1 \partial x^n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial^2 f}{\partial x^n \partial x^1} & \dfrac{\partial^2 f}{\partial x^n \partial x^2} & \cdots & \dfrac{\partial^2 f}{\partial x^n \partial x^n} \end{bmatrix}.$$

HINT: Apply Proposition 2.3.7. Recall that the matrix of a bilinear mapping $B \in L(\mathbb{R}^n, \mathbb{R}^m; \mathbb{R})$ has the entries $B(e_i, f_j)$ (first index = row index, second index = column index), where $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$ are ordered bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

2.4 Properties of the Derivative

In this section some of the fundamental properties of the derivative are developed. These properties are analogues of rules familiar from elementary calculus.

Differentiability implies Lipschitz. Let us begin by strengthening the fact that differentiability implies continuity.

2.4.1 Proposition. *Suppose $U \subset \mathbf{E}$ is open and $f : U \to \mathbf{F}$ is differentiable on U. Then f is continuous. In fact, for each $u_0 \in U$ there is a constant $M > 0$ and a $\delta_0 > 0$ with the property that $\|u - u_0\| < \delta_0$ implies $\|f(u) - f(u_0)\| \leq M\|u - u_0\|$. (This is called the **Lipschitz property**.)*

Proof. Using the general inequality $|\|e_1\| - \|e_2\|| \leq \|e_1 - e_2\|$, we get

$$\begin{aligned}|\,\|f(u) - f(u_0)\| - \|\mathbf{D}f(u_0)\cdot(u-u_0)\|\,| \\ &\leq \|f(u) - f(u_0) - \mathbf{D}f(u_0)\cdot(u-u_0)\| \\ &= \|o(u-u_0)\| \leq \|u-u_0\|\end{aligned}$$

for $\|u - u_0\| \leq \delta_0$, where δ_0 is some positive constant depending on u_0; this holds since

$$\lim_{u\to u_0} \frac{o(u-u_0)}{\|u-u_0\|} = 0.$$

Thus,

$$\begin{aligned}\|f(u) - f(u_0)\| &\leq \|\mathbf{D}f(u_0)\cdot(u-u_0)\| + \|u-u_0\| \\ &\leq (\|\mathbf{D}f(u_0)\| + 1)\|u-u_0\|\end{aligned}$$

for $\|u - u_0\| \leq \delta_0$. ■

Chain Rule. Perhaps the most important rule of differential calculus is the chain rule. To facilitate its statement, the notion of the tangent of a map is introduced. The text will begin conceptually distinguishing *points* in U from *vectors* in $\mathbf{E}$. At this point it is not so clear that the distinction is important, but it will help with the transition to manifolds in Chapter 3.

2.4.2 Definition. *Suppose $f : U \subset \mathbf{E} \to \mathbf{F}$ is of class C^1. Define the* ***tangent*** *of f to be the map*

$$Tf : U \times \mathbf{E} \to \mathbf{F} \times \mathbf{F} \quad \text{given by } Tf(u,e) = (f(u), \mathbf{D}f(u)\cdot e),$$

where we recall that $\mathbf{D}f(u)\cdot e$ denotes $\mathbf{D}f(u)$ applied to $e \in \mathbf{E}$ as a linear map. If f is of class C^r, define $T^r f = T(T^{r-1}f)$ inductively.

From a geometric point of view, Tf is a more "natural" object than D. The reasons for this will become clearer as we proceed, but roughly speaking, the essence is this: we think of (u,e) as a vector with base point u, and vector part e then $(f(u), \mathbf{D}f(u)\cdot e)$ is the image vector with its base point $f(u)$, as in Figure 2.4.1. Another reason for this is the simple and elegant behavior of T under composition, as given in the next theorem.

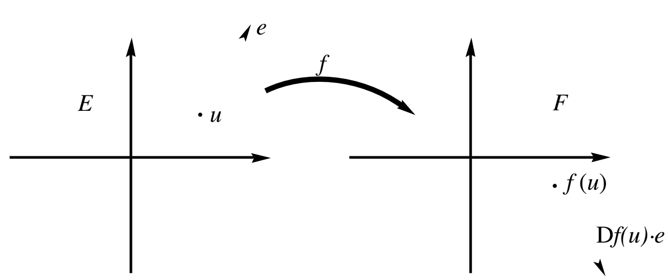

FIGURE 2.4.1. The geometry of the tangent map

2.4.3 Theorem (C^r Composite Mapping Theorem)**.** *Suppose $f : U \subset \mathbf{E} \to V \subset \mathbf{F}$ and $g : V \subset \mathbf{F} \to \mathbf{G}$ are differentiable (resp., C^r) maps. Then the composite $g \circ f : U \subset \mathbf{E} \to \mathbf{G}$ is also differentiable (resp., C^r) and*

$$T(g \circ f) = Tg \circ Tf,$$

(resp., $T^r(g \circ f) = T^r g \circ T^r f$). The formula $T(g \circ f) = Tg \circ Tf$ is equivalent to the chain rule in terms of $\mathbf{D}$*:*

$$\mathbf{D}(g \circ f)(u) = \mathbf{D}g(f(u)) \circ \mathbf{D}f(u).$$

Proof. Since f is differentiable at $u \in U$ and g is differentiable at $f(u) \in V$, we have

$$f(u+e) = f(u) + \mathbf{D}f(u) \cdot e + o(e) \quad \text{for } e \in \mathbf{E}$$

and for $v = f(u)$ we have $g(v+w) = g(v) + \mathbf{D}g(v) \cdot w + o(w)$. Thus,

$$\begin{aligned}(g \circ f)(u+e) &= g(f(u) + \mathbf{D}f(u) \cdot e + o(e)) \\ &= (g \circ f)(u) + \mathbf{D}g(f(u)) \cdot (\mathbf{D}f(u) \cdot e) \\ &\quad + \mathbf{D}g(f(u))(o(e)) + o(\mathbf{D}f(u) \cdot e + o(e)).\end{aligned}$$

For e in a neighborhood of the origin,

$$\frac{\|\mathbf{D}f(u) \cdot e + o(e)\|}{\|e\|} \leq \left(\|\mathbf{D}f(u)\| + \frac{\|o(e)\|}{\|e\|} \right) \leq M$$

for some constant $M > 0$, and

$$\|\mathbf{D}g(f(u)) \cdot o(e)\| \leq \|\mathbf{D}g(f(u))\| \, \|o(e)\|.$$

Therefore,

$$\begin{aligned}\frac{\|o(\mathbf{D}f(u) \cdot e + o(e))\|}{\|e\|} &= \frac{\|(o(\mathbf{D}f(u) \cdot e + o(e)))\|}{\|\mathbf{D}f(u) \cdot e + o(e)\|} \cdot \frac{\|\mathbf{D}f(u) \cdot e + o(e)\|}{\|e\|} \\ &\leq M \frac{\|(o(\mathbf{D}f(u) \cdot e + o(e)))\|}{\|\mathbf{D}f(u) \cdot e + o(e)\|}.\end{aligned}$$

Hence, we conclude that

$$\mathbf{D}g(f(u)) \cdot (o(e)) + o(\mathbf{D}f(u) \cdot e + o(e)) = o(e)$$

and thus

$$\mathbf{D}(g \circ f)(u) \cdot e = \mathbf{D}g(f(u)) \cdot (\mathbf{D}f(u) \cdot e).$$

Denote by $\varphi : L(\mathbf{F}, \mathbf{G}) \times L(\mathbf{E}, \mathbf{F}) \to L(\mathbf{E}, \mathbf{G})$ the bilinear mapping $\varphi(B, A) = B \circ A$ and note that $\varphi \in L(L(\mathbf{F}, \mathbf{G}), L(\mathbf{E}, \mathbf{F}); L(\mathbf{E}, \mathbf{G}))$ since $\|B \circ A\| \leq \|B\| \, \|A\|$; that is, $\|\varphi\| \leq 1$. Let $(\mathbf{D}g \circ f) \times \mathbf{D}f : U \to L(\mathbf{F}, \mathbf{G}) \times L(\mathbf{E}, \mathbf{F})$ be defined by

$$[(\mathbf{D}g \circ f) \times \mathbf{D}f](u) = (\mathbf{D}g(f(u)), \mathbf{D}f(u));$$

notice that this map is continuous if f and g are of class C^1. Therefore the composite function

$$\varphi \circ ((\mathbf{D}g \circ f) \times \mathbf{D}f) = \mathbf{D}(g \circ f) : U \to L(\mathbf{E}, \mathbf{G})$$

is continuous if f and g are C^1, that is, $g \circ f$ is C^1. Inductively suppose f and g are C^r. Then $\mathbf{D}g$ is C^{r-1}, so $\mathbf{D}g \circ f$ is C^{r-1} and thus the map $(\mathbf{D}g \circ f) \times \mathbf{D}f$ is C^{r-1} (see Proposition 2.3.6). Since φ is C^∞ (Exercise 2.3-1), again the inductive hypothesis forces $\varphi \circ ((\mathbf{D}g \circ f) \times \mathbf{D}f) = \mathbf{D}(g \circ f)$ to be C^{r-1}; that is, $g \circ f$ is C^r.

The formula $T^r(g \circ f) = T^r g \circ T^r f$ is a direct verification for $r = 1$ using the chain rule, and the rest follows by induction. ■

If $\mathbf{E} = \mathbb{R}^m$, $\mathbf{F} = \mathbb{R}^n$, $\mathbf{G} = \mathbb{R}^p$, and $f = (f^1, \ldots, f^n)$, $g = (g^1, \ldots, g^p)$, where $f^i : U \to \mathbb{R}$ and $g^j : V \to \mathbb{R}$, by Proposition 2.3.7 the chain rule becomes

$$\begin{bmatrix} \dfrac{\partial(g\circ f)^1(x)}{\partial x^1} & \cdots & \dfrac{\partial(g\circ f)^1(x)}{\partial x^m} \\ \vdots & & \vdots \\ \dfrac{\partial(g\circ f)^p(x)}{\partial x^1} & \cdots & \dfrac{\partial(g\circ f)^p(x)}{\partial x^m} \end{bmatrix}$$
$$= \begin{bmatrix} \dfrac{\partial g^1(f(x))}{\partial y^1} & \cdots & \dfrac{\partial g^1(f(x))}{\partial y^n} \\ \vdots & & \vdots \\ \dfrac{\partial g^p(f(x))}{\partial y^1} & \cdots & \dfrac{\partial g^p(f(x))}{\partial y^n} \end{bmatrix} \cdot \begin{bmatrix} \dfrac{\partial f^1(x)}{\partial x^1} & \cdots & \dfrac{\partial f^1(x)}{\partial x^m} \\ \vdots & & \vdots \\ \dfrac{\partial f^n(x)}{\partial x^1} & \cdots & \dfrac{\partial f^n(x)}{\partial x^m} \end{bmatrix}$$

which, when read componentwise, becomes the usual chain rule from calculus:

$$\frac{\partial(g\circ f)^j(x)}{\partial x^i} = \sum_{k=1}^{n} \frac{\partial g^j(f(x))}{\partial y^k}\frac{\partial f^k(x)}{\partial x^i}, \quad i = 1, \ldots, m.$$

Product Rule. The chain rule applied to $B \in L(\mathbf{F}_1, \mathbf{F}_2; \mathbf{G})$ and $f_1 \times f_2 : U \subset \mathbf{E} \to \mathbf{F}_1 \times \mathbf{F}_2$ yields the following.

2.4.4 Theorem (The Leibniz or Product Rule). *Let $f_i : U \subset \mathbf{E} \to \mathbf{F}_i$, $i = 1, 2$, be differentiable (resp., C^r) maps and $B \in L(\mathbf{F}_1, \mathbf{F}_2; \mathbf{G})$. Then the mapping $B(f_1, f_2) = B \circ (f_1 \times f_2) : U \subset \mathbf{E} \to \mathbf{G}$ is differentiable (resp., C^r) and*

$$\mathbf{D}(B(f_1, f_2))(u) \cdot e = B(\mathbf{D}f_1(u) \cdot e, f_2(u)) + B(f_1(u), \mathbf{D}f_2(u) \cdot e).$$

In the case $\mathbf{F}_1 = \mathbf{F}_2 = \mathbb{R}$ and B is multiplication, Theorem 2.4.4 reduces to the usual product rule for derivatives. Leibniz' rule can easily be extended to multilinear mappings (Exercise 2.4-3).

Directional Derivatives. The first of several consequences of the chain rule involves the directional derivative.

2.4.5 Definition. *Let $f : U \subset \mathbf{E} \to \mathbf{F}$ and let $u \in U$. We say that f **has a derivative in the direction** $e \in \mathbf{E}$ at u if*

$$\left.\frac{d}{dt} f(u + te)\right|_{t=0}$$

*exists. We call this element of $\mathbf{F}$ **the directional derivative of** f **in the direction** e at u.*

Sometimes a function all of whose directional derivatives exist is called ***Gâteaux differentiable***, whereas a function differentiable in the sense we have defined is called ***Fréchet differentiable***. The latter is stronger, according to the following. (See also Exercise 2.4-10.)

2.4.6 Proposition. *If f is differentiable at u, then the directional derivatives of f exist at u and are given by*

$$\left.\frac{d}{dt} f(u + te)\right|_{t=0} = \mathbf{D}f(u) \cdot e.$$

Proof. A ***path*** in $\mathbf{E}$ is a map from I into $\mathbf{E}$, where I is an open interval of $\mathbb{R}$. Thus, if c is differentiable, for $t \in I$ we have $\mathbf{D}c(t) \in L(\mathbb{R}, \mathbf{E})$, by definition. Recall that we identify $L(\mathbb{R}, \mathbf{E})$ with $\mathbf{E}$ by associating $\mathbf{D}c(t)$ with $\mathbf{D}c(t) \cdot 1$ ($1 \in \mathbb{R}$). Let

$$\frac{dc}{dt}(t) = \mathbf{D}c(t) \cdot 1.$$

For $f : U \subset \mathbf{E} \to \mathbf{F}$ of class C^1 we consider $f \circ c$, where $c : I \to U$. It follows from the chain rule that

$$\frac{d}{dt}(f(c(t))) = \mathbf{D}(f \circ c)(t) \cdot 1 = \mathbf{D}f(c(t)) \cdot \frac{dc}{dt}$$

The proposition follows by choosing $c(t) = u + te$, where $u, e \in \mathbf{E}$, $I =]-\lambda, \lambda[$, and λ is sufficiently small. ■

For $f : U \subset \mathbb{R}^n \to \mathbb{R}$, the directional derivative is given in terms of the standard basis $\{e_1, \ldots, e_n\}$ by

$$\mathbf{D}f(u) \cdot e = \frac{\partial f}{\partial x^1} x^1 + \cdots + \frac{\partial f}{\partial x^n} x^n,$$

where $e = x^1 e_1 + \cdots + x^n e_n$. This follows from Proposition 2.3.7 and Proposition 2.4.6.

The formula in Proposition 2.4.6 is sometimes a convenient method for computing $\mathbf{D}f(u) \cdot e$. For example, let us compute the differential of a homogeneous polynomial of degree 2 from $\mathbf{E}$ to $\mathbf{F}$. Let $f(e) = A(e, e)$, where $A \in L^2(\mathbf{E}; \mathbf{F})$. By the chain and Leibniz rules,

$$\mathbf{D}f(u) \cdot e = \frac{d}{dt} A(u + te, u + te)\bigg|_{t=0} = A(u, e) + A(e, u).$$

If A is symmetric, then $\mathbf{D}f(u) \cdot e = 2A(u, e)$.

Mean Value Inequality. One of the basic tools for finding estimates is the following.

2.4.7 Proposition. *Let $\mathbf{E}$ and $\mathbf{F}$ be real Banach spaces, $f : U \subset \mathbf{E} \to \mathbf{F}$ a C^1-map, $x, y \in U$, and c a C^1 arc in U connecting x to y; that is, c is a continuous map $c : [0, 1] \to U$, which is C^1 on $]0, 1[$, $c(0) = x$, and $c(1) = y$. Then*

$$f(y) - f(x) = \int_0^1 \mathbf{D}f(c(t)) \cdot c'(t)\, dt.$$

If U is convex and $c(t) = (1 - t)x + ty$, then

$$\begin{aligned} f(y) - f(x) &= \int_0^1 \mathbf{D}f((1 - t)x + ty) \cdot (y - x)\, dt \\ &= \left(\int_0^1 \mathbf{D}f((1 - t)x + ty)\, dt \right) \cdot (y - x). \end{aligned}$$

Proof. If $g(t) = (f \circ c)(t)$, the chain rule implies $g'(t) = \mathbf{D}f(c(t)) \cdot c'(t)$ and the fundamental theorem of calculus gives

$$g(1) - g(0) = \int_0^1 g'(t)\, dt,$$

which is the first equality. The second equality for U convex and $c(t) = (1 - t)x + ty$ is Exercise 2.2-6(i). ■

2.4.8 Proposition (Mean Value Inequality). *Suppose $U \subset \mathbf{E}$ is convex and $f : U \subset \mathbf{E} \to \mathbf{F}$ is C^1. Then for all $x, y \in U$*

$$\|f(y) - f(x)\| \le \left[\sup_{0 \le t \le 1} \|\mathbf{D}f((1-t)x + ty)\| \right] \|y - x\|.$$

Thus, if $\|\mathbf{D}f(u)\|$ is uniformly bounded on U by a constant $M > 0$, then for all $x, y \in U$

$$\|f(y) - f(x)\| \le M\|y - x\|.$$

If $\mathbf{F} = \mathbb{R}$, then $f(y) - f(x) = \mathbf{D}f(c) \cdot (y - x)$ for some c on the line joining x to y.

Proof. The inequality follows directly from Proposition 2.4.7. The last assertion follows from the intermediate value theorem as in elementary calculus. ■

2.4.9 Corollary. *Let $U \subset \mathbf{E}$ be an open set; then the following are equivalent:*

(i) *U is connected*;

(ii) *every differentiable map $f : U \subset \mathbf{E} \to \mathbf{F}$ satisfying $\mathbf{D}f = 0$ on U is constant.*

Proof. If $U = U_1 \cup U_2$ and $U_1 \cap U_2 = \varnothing$, where U_1 and U_2 are open, then the mapping

$$f(u) = \begin{cases} 0, & \text{if } u \in U_1; \\ e, & \text{if } u \in U_2, \end{cases}$$

where $e \in \mathbf{F}$, $e \neq 0$ is a fixed vector, has $\mathbf{D}f = 0$, yet is not constant.

Conversely, assume that U is connected and $\mathbf{D}f = 0$. Then f is in fact C^∞. Let $u_0 \in U$ be fixed and consider the set $S = \{ u \in U \mid f(u) = f(u_0) \}$. Then $S \neq \varnothing$ (since $u_0 \in S$), $S \subset U$, and S is closed since f is continuous. We shall show that S is also open. If $u \in S$, consider $v \in D_r(u) \subset U$ and apply Proposition 2.4.8 to get

$$\|f(u) - f(v)\| \le \sup\{ \|\mathbf{D}f((1-t)u + tv)\| \mid t \in [0,1] \} \|u - v\| = 0;$$

that is, $f(v) = f(u) = f(u_0)$ and hence $D_r(u) \subset S$. Connectedness of U implies $S = U$. ■

If f is Gâteaux differentiable and the Gâteaux derivative is in $L(\mathbf{E}, \mathbf{F})$; that is, for each $u \in V$ there exists $G_u \in L(\mathbf{E}, \mathbf{F})$ such that

$$\left. \frac{d}{dt} f(u + te) \right|_{t=0} = G_u e,$$

and if $u \mapsto G_u$ is continuous, we say f is ***C^1-Gâteaux***. The mean value inequality holds, replacing C^1 everywhere by "C^1-Gâteaux" and the identical proofs work. When studying differentiability the following is often useful.

2.4.10 Corollary. *If $f : U \subset \mathbf{E} \to \mathbf{F}$ is C^1-Gâteaux then it is C^1 and the two derivatives coincide.*

Proof. Let $u \in U$ and work in a disk centered at u. Proposition 2.4.7 gives

$$\begin{aligned} \|f(u + e) - f(u) - G_u e\| &= \left\| \left(\int_0^1 (G_{u+te} - G_u)\, dt \right) e \right\| \\ &\le \sup\{ \|G_{u+te} - G_u\| \mid t \in [0, a] \} \|e\| \end{aligned}$$

and the sup converges to zero as, $e \to 0$, by uniform continuity of the map $t \in [0,1] \mapsto G_{u+te} \in L(\mathbf{E}, \mathbf{F})$. This says that $\mathbf{D}f(u) \cdot e$ exists and equals $G_u e$. ■

Partial Derivatives. We shall discuss only functions of two variables, the generalization to n variables being obvious.

2.4.11 Definition. *Let $f : U \to \mathbf{F}$ be a mapping defined on the open set $U \subset \mathbf{E}_1 \oplus \mathbf{E}_2$ and let $u_0 = (u_{01}, u_{02}) \in U$. The derivatives of the mappings $v_1 \mapsto f(v_1, u_{02})$, $v_2 \mapsto f(u_{01}, v_2)$, where $v_1 \in \mathbf{E}_1$ and $v_2 \in \mathbf{E}_2$, if they exist, are called **partial derivatives** of f at $u_0 \in U$ and are denoted by $\mathbf{D}_1 f(u_0) \in L(\mathbf{E}_1, \mathbf{F})$, $\mathbf{D}_2 f(u_0) \in L(\mathbf{E}_2, \mathbf{F})$.*

2.4.12 Proposition. *Let $U \subset \mathbf{E}_1 \oplus \mathbf{E}_2$ be open and $f : U \to \mathbf{F}$.*

(i) *If f is differentiable, then the partial derivatives exist and are given by*

$$\mathbf{D}_1 f(u) \cdot e_1 = \mathbf{D} f(u) \cdot (e_1, 0) \quad \textit{and} \quad \mathbf{D}_2 f(u) \cdot e_2 = \mathbf{D} f(u) \cdot (0, e_2).$$

(ii) *If f is differentiable, then*

$$\mathbf{D} f(u) \cdot (e_1, e_2) = \mathbf{D}_1 f(u) \cdot e_1 + \mathbf{D}_2 f(u) \cdot e_2.$$

(iii) *f is of class C^r iff $\mathbf{D}_i f : U \to L(\mathbf{E}_i, \mathbf{F})$, $i = 1, 2$ both exist and are of class C^{r-1}.*

Proof. (i) Let $j_u^1 : \mathbf{E}_1 \to \mathbf{E}_1 \oplus \mathbf{E}_2$ be defined by $j_u^1(v_1) = (v_1, u_2)$, where $u = (u_1, u_2)$. Then j_u^1 is C^∞ and $\mathbf{D} j_u^1(u_1) = J_1 \in L(\mathbf{E}_1, \mathbf{E}_1 \oplus \mathbf{E}_2)$ is given by $J_1(e_1) = (e_1, 0)$. By the chain rule,

$$\mathbf{D}_1 f(u) = \mathbf{D}(f \circ j_u^1)(u_1) = \mathbf{D} f(u) \cdot J_1,$$

which proves the first relation in (i). One similarly defines j_u^2, J_2, and proves the second relation.

(ii) Let $P_i(e_1, e_2) = e_i$, $i = 1, 2$ be the canonical projections. Then compose the relation $J_1 \circ P_1 + J_2 \circ P_2 =$ identity on $\mathbf{E}_1 \oplus \mathbf{E}_2$ with $\mathbf{D} f(u)$ on the left and use (i).

(iii) Let

$$\Phi_i \in L(L(\mathbf{E}_1 \oplus \mathbf{E}_2, \mathbf{F}), L(\mathbf{E}_i, \mathbf{F}))$$

and

$$\Psi_i \in L(L(\mathbf{E}_i, \mathbf{F}), L(\mathbf{E}_1 \oplus \mathbf{E}_2, \mathbf{F}))$$

be defined by $\Phi_i(A) = A \circ J_i$ and $\Psi_i(B_i) = B_i \circ P_i$, $i = 1, 2$. Then (i) and (ii) become

$$\mathbf{D}_i f = \Phi_i \circ \mathbf{D} f \qquad \mathbf{D} f = \Psi_1 \circ \mathbf{D}_1 f + \Psi_2 \circ \mathbf{D}_2 f.$$

This shows that if f is differentiable, then f is C^r iff $\mathbf{D}_1 f$ and $\mathbf{D}_2 f$ are C^{r-1}. Thus to conclude the proof we need to show that if $\mathbf{D}_1 f$ and $\mathbf{D}_2 f$ exist and are continuous, then $\mathbf{D} f$ exists. By Proposition 2.4.7 applied consecutively to the two arguments, we get

$$\begin{aligned} f(u_1 + e_1, u_2 + e_2) - f(u_1, u_2) - \mathbf{D}_1 f(u_1, u_2) \cdot e_1 - \mathbf{D}_2 f(u_1, u_2) \cdot e_2 \\ = f(u_1 + e_1, u_2 + e_2) - f(u_1, u_2 + e_2) - \mathbf{D}_1 f(u_1, u_2) \cdot e_1 \\ + f(u_1, u_2 + e_2) - f(u_1, u_2) - \mathbf{D}_2 f(u_1, u_2) \cdot e_2 \\ = \left(\int_0^1 (\mathbf{D}_1 f(u_1 + te_1, u_2 + e_2) - \mathbf{D}_1 f(u_1, u_2)) \, dt \right) \cdot e_1 \\ + \left(\int_0^1 (\mathbf{D}_2 f(u_1, u_2 + te_2) - \mathbf{D}_2 f(u_1, u_2)) \, dt \right) \cdot e_2 \end{aligned}$$

Taking norms and using in each term the obvious inequality $\|e_1\| \le \|e_1\| + \|e_2\| \equiv \|(e_1, e_2)\|$, we see that

$$\begin{aligned}
&\|f(u_1 + e_1, u_2 + e_2) - f(u_1, u_2) - \mathbf{D}_1 f(u_1, u_2) \cdot e_1 - \mathbf{D}_2 f(u_1, u_2) \cdot e_2\| \\
&\quad \le \left(\sup_{0 \le t \le 1} \|\mathbf{D}_1 f(u_1 + te_1, u_2 + e_2) - \mathbf{D}_1 f(u_1, u_2 + e_2)\| \right. \\
&\qquad \left. + \sup_{0 \le t \le 1} \|\mathbf{D}_2 f(u_1, u_2 + te_2) - \mathbf{D}_2 f(u_1, u_2)\| \right) \|(e_1, e_2)\|.
\end{aligned}$$

Both sups in the parentheses converge to zero as $(e_1, e_2) \to (0, 0)$ by continuity of the partial derivatives. ■

Higher Derivatives. If $\mathbf{E}_1 = \mathbf{E}_2 = \mathbb{R}$ and $\{e_1, e_2\}$ is the standard basis in $\mathbb{R}^2$ we see that

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} = \mathbf{D}_1 f(x, y) \cdot e_1 \in \mathbf{F}.$$

Similarly, $(\partial f / \partial y)(x, y) = \mathbf{D}_2 f(x, y) \cdot e_2 \in \mathbf{F}$. Define inductively higher derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \quad \text{etc.}$$

2.4.13 Example. As an application of the formalism just introduced we shall prove that for $f : U \subset \mathbb{R}^2 \to \mathbb{R}$

$$\begin{aligned}
\mathbf{D}^2 f(u) \cdot (v, w) &= v^1 w^1 \frac{\partial^2 f}{\partial x^2}(u) + v^1 w^2 \frac{\partial^2 f}{\partial y \partial x}(u) + v^2 w^1 \frac{\partial^2 f}{\partial x \partial y}(u) \\
&\quad + v^2 w^2 \frac{\partial^2 f}{\partial y^2}(u), \\
&= (v^1, v^2) \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2}(u) & \dfrac{\partial^2 f}{\partial y\, \partial x}(u) \\ \dfrac{\partial^2 f}{\partial x\, \partial y}(u) & \dfrac{\partial^2 f}{\partial y^2}(u) \end{bmatrix} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix},
\end{aligned}$$

where $u \in U$, $v, w \in \mathbb{R}^2$, $v = v^1 e_1 + v^2 e_2$, $w = w^1 e_1 + w^2 e_2$, and $\{e_1, e_2\}$ is the standard basis of $\mathbb{R}^2$. To prove this, note that by definition,

$$\mathbf{D}^2 f(u) \cdot (v, w) = \mathbf{D}((\mathbf{D} f)(\cdot) \cdot w)(u) \cdot v.$$

Applying the chain rule to $\mathbf{D} f(\cdot) \cdot w = T_w : A \in L(\mathbb{R}^2, \mathbf{F}) \mapsto A \cdot w \in \mathbf{F}$, the above

$$\begin{aligned}
&\mathbf{D}(\mathbf{D} f(\cdot) \cdot w)(u) \cdot v \\
&\quad = \mathbf{D}(\mathbf{D}_1 f(\cdot) \cdot w^1 e_1 + \mathbf{D}_2 f(\cdot) \cdot w^2 e_2)(u) \cdot v \qquad \text{(by Prop. 2.4.12(ii))} \\
&\quad = \mathbf{D}\left(w^1 \frac{\partial f}{\partial x} + w^2 \frac{\partial f}{\partial y}\right)(u) \cdot v \\
&\quad = w^1 \left[\mathbf{D}_1 \left(\frac{\partial f}{\partial x}\right)(u) \cdot v^1 e_1 + \mathbf{D}_2 \left(\frac{\partial f}{\partial x}\right)(u) \cdot v^2 e_2\right] \qquad (2.4.1) \\
&\qquad + w^2 \left[\mathbf{D}_1 \left(\frac{\partial f}{\partial y}\right)(u) \cdot v^1 e_1 + \mathbf{D}_2 \left(\frac{\partial f}{\partial y}\right)(u) \cdot v^2 e_2\right] \\
&\quad = v^1 w^1 \frac{\partial^2 f}{\partial x^2}(u) + v^2 w^1 \frac{\partial^2 f}{\partial x \partial y}(u) + v^1 w^2 \frac{\partial^2 f}{\partial y \partial x}(u) + v^2 w^2 \frac{\partial^2 f}{\partial y^2}(u).
\end{aligned}$$

♦

For computation of higher derivatives, note that by repeated application of Proposition 2.4.6,

$$\mathbf{D}^r f(u) \cdot (e_1, \ldots, e_r) = \frac{d}{dt_r} \cdots \frac{d}{dt_1} \left\{ f\left(u + \sum_{i=1}^{r} t_i e_i \right) \right\} \Bigg|_{t_1 = \cdots = t_r = 0}$$

In particular, for $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ the components of $\mathbf{D}^r f(u)$ in terms of the standard basis are

$$\frac{\partial^r f}{\partial x^{i_1} \cdots \partial x^{i_r}}, \quad 0 \leq i_k \leq r.$$

Thus, f is of class C^r iff all its r-th order partial derivatives exist and are continuous.

Symmetry of Higher Derivatives. Equality of mixed partials is of course a fundamental property we learn in calculus. Here is the general result.

2.4.14 Proposition (L. Euler). *If $f : U \subset \mathbf{E} \to \mathbf{F}$ is C^r, then $\mathbf{D}^r f(u) \in L_s^r(\mathbf{E}, \mathbf{F})$; that is, $\mathbf{D}^r f(u)$ is symmetric.*

Proof. First we prove the result for $r = 2$. Let $u \in U$, $v, w \in \mathbf{E}$ be fixed; we want to show that $\mathbf{D}^2 f(u) \cdot (v, w) = \mathbf{D}^2 f(u) \cdot (w, v)$. To this, define the linear map $a : \mathbb{R}^2 \to \mathbf{E}$ by $a(e_1) = v$, and $a(e_2) = w$, where e_1 and e_2 are the standard basis vectors of $\mathbb{R}^2$. For $(x, y) \in \mathbb{R}^2$, then $a(x, y) = xv + yw$. Now define the affine map $A : \mathbb{R}^2 \to \mathbf{E}$ by $A(x, y) = u + a(x, y)$. Since

$$\mathbf{D}^2 (f \circ A)(x, y) \cdot (e_1, e_2) = \mathbf{D}^2 f(u) \cdot (v, w)$$

(Exercise 2.3-5), it suffices to prove this formula:

$$\mathbf{D}^2 (f \circ A) \cdot (x, y) \cdot (e_1, e_2) = \mathbf{D}^2 (f \circ A)(x, y) \cdot (e_2, e_1);$$

that is,

$$\frac{\partial^2 (f \circ A)}{\partial x \partial y} = \frac{\partial^2 (f \circ A)}{\partial y \partial x}$$

(see Example 2.4.13). Let $g = f \circ A : V = A^{-1}(U) \subset \mathbb{R}^2 \to \mathbf{F}$. Since for any $\lambda \in \mathbf{F}^*$, $\partial^2 (\lambda \circ g)/\partial x \partial y = \lambda(\partial^2 g/\partial x \partial y)$, using the Hahn–Banach theorem 2.2.12, it suffices to prove that

$$\frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial^2 \varphi}{\partial y \partial x},$$

where $\varphi = \lambda \circ g : V \subset \mathbb{R}^2 \to \mathbb{R}$, which is a standard result from calculus. For the sake of completeness we recall the proof. Applying the mean value theorem twice, we get

$$\begin{aligned} S_{h,k} &= [\varphi(x + h, y + k) - \varphi(x, y + k)] - [\varphi(x + h, y) - \varphi(x, y)] \\ &= \left(\frac{\partial \varphi}{\partial x}(c_{h,k}, y + k) = \frac{\partial \varphi}{\partial x}(c_{h,k}, y) \right) k \\ &= \frac{\partial^2 \varphi}{\partial x \partial y}(c_{h,k}, d_{h,k}) hk. \end{aligned}$$

for some $c_{h,k}, d_{h,k}$ lying between x and $x + h$, and y and $y + k$, respectively. By interchanging the two middle terms in $S_{h,k}$ we can derive in the same way that

$$S_{h,k} = \frac{\partial^2 \varphi}{\partial y \partial x}(\gamma_{h,k}, \delta_{h,k}) hk.$$

Equating these two formulas for $S_{h,k}$, canceling h, k, and letting $h \to 0$, $k \to 0$, the continuity of $\mathbf{D}^2\varphi$ gives the result.

For general r, proceed by induction:

$$\begin{aligned}\mathbf{D}^r f(u) \cdot (v_1, v_2, \ldots, v_n) &= \mathbf{D}^2(\mathbf{D}^{r-2} f)(u) \cdot (v_1, v_2) \cdot (v_3, \ldots, v_n)\\ &= \mathbf{D}^2(\mathbf{D}^{r-2} f)(u) \cdot (v_2, v_1) \cdot (v_3, \ldots, v_n)\\ &= \mathbf{D}^r f(u) \cdot (v_2, v_1, v_3, \ldots, v_n).\end{aligned}$$

Let σ be any permutation of $\{2, \ldots, n\}$, so by the inductive hypothesis

$$\mathbf{D}^{r-1} f(u)(v_2, \ldots, v_n) = \mathbf{D}^{r-1} f(u)(v_{\sigma(2)}, \ldots, v_{\sigma(n)}).$$

Take the derivative of this relation with respect to $u \in U$ keeping $v_2, \ldots, v_n$ fixed and get (Exercise 2.4-6):

$$\mathbf{D}^r f(u)(v_1, \ldots, v_n) = \mathbf{D}^r f(u)(v_1, v_{\sigma(2)}, \ldots, v_{\sigma(n)}).$$

Since any permutation can be written as a product of the transposition $\{1, 2, 3, \ldots, n\} \to \{2, 1, 3, \ldots, n\}$ (if necessary) and a permutation of the set $\{2, \ldots, n\}$, the result follows. ■

Taylor's Theorem. Suppose $U \subset \mathbf{E}$ is an open set. Since $+ : \mathbf{E} \times \mathbf{E} \to \mathbf{E}$ is continuous, there exists an open set $\tilde{U} \subset \mathbf{E} \times \mathbf{E}$ with these three properties:

(i) $U \times \{0\} \subset \tilde{U}$,

(ii) $u + \xi h \in U$ for all $(u, h) \in \tilde{U}$ and $0 \le \xi \le 1$, and

(iii) $(u, h) \in \tilde{U}$ implies $u \in U$.

For example let

$$\tilde{U} = \{(+)^{-1}(U)\} \cap (U \times \mathbf{E}).$$

Let us call such a set $\tilde{U}$ a ***thickening*** of U. See Figure 2.4.2.

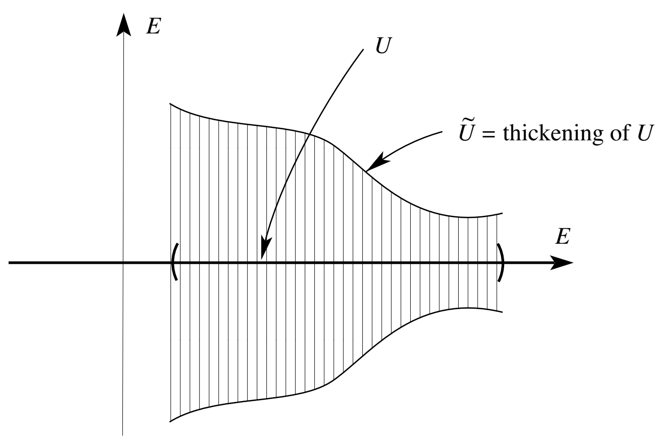

FIGURE 2.4.2. A thickened neighborhood

2.4.15 Theorem (Taylor's Theorem). *A map $f : U \subset \mathbf{E} \to \mathbf{F}$ is of class C^r iff there are continuous mappings*

$$\varphi_p : U \subset \mathbf{E} \to L_s^p(\mathbf{E}, \mathbf{F}), \quad p = 1, \ldots, r, \quad \textit{and} \quad R : \tilde{U} \to L_s^r(\mathbf{E}, \mathbf{F}),$$

where $\tilde{U}$ is some thickening of U such that for all $(u,h) \in \tilde{U}$,

$$f(u+h) = f(u) + \frac{\varphi_1(u)}{1!} \cdot h + \frac{\varphi_2(u)}{2!} \cdot h^2 + \cdots + \frac{\varphi_r(u)}{r!} \cdot h^r + R(u,h) \cdot h^r,$$

where $h^p = (h, \ldots, h)$ (p times) and $R(u,0) = 0$. If f is C^r then necessarily $\varphi_p = \mathbf{D}^p f$ and

$$R(u,h) = \int_0^1 \frac{(1-t)^{r-1}}{(r-1)!} (\mathbf{D}^r f(u+th) - \mathbf{D}^r f(u))\, dt.$$

Proof. We shall prove the "only if" part. The converse is proved in Supplement 2.4B. Leibniz' rule gives the following ***integration by parts*** formula. If $[a,b] \subset U \subset \mathbb{R}$ and $\psi_i : U \subset \mathbb{R} \to \mathbf{E}_i$, $i = 1,2$ are C^1 mappings and $B \in L(\mathbf{E}_1, \mathbf{E}_2; \mathbf{F})$ is a bilinear map of $\mathbf{E}_1 \times \mathbf{E}_2$ to $\mathbf{F}$, then

$$\begin{aligned} \int_a^b B(\psi_1'(1), \psi_2(t))\, dt &= B(\psi_1(b), \psi_2(b)) - B(\psi_1(a), \psi_2(a)) \\ &\qquad - \int_a^b B(\psi_1(t), \psi_2'(t))\, dt. \end{aligned}$$

Assume f is a C^r mapping. If $r = 1$, then by Proposition 2.4.7

$$\begin{aligned} f(u+h) &= f(u) + \left(\int_0^1 \mathbf{D}f(u+th)\, dt \right) \cdot h \\ &= f(u) + \mathbf{D}f(u) \cdot h + \left(\int_0^1 (\mathbf{D}f(u+th) - \mathbf{D}f(u))\, dt \right) \cdot h \end{aligned}$$

and the formula is proved. For general $k \le r$ proceed by induction choosing in the integration by parts formula $\mathbf{E}_1 = \mathbb{R}$, $\mathbf{E}_2 = \mathbf{E}$, $B(s,e) = se$, $\psi_2(t) = \mathbf{D}^k f(u+th) \cdot h^k$, and $\psi_1(t) = -(1-t)^k/k!$, and taking into account that

$$\int_0^1 \frac{(1-t)^k}{k!}\, dt = \frac{1}{(k+1)!}.$$

Since $\mathbf{D}^k f(u) \in L_s^k(\mathbf{E}, \mathbf{F})$ by Proposition 2.4.14, Taylor's formula follows. ■

Note that $R(u,h) \cdot h^r = o(h^r)$ since $R(u,h) \to 0$ as $h \to 0$. If f is C^{r+1} then the mean value inequality and a bound on $\mathbf{D}^{r+1} f$ gives $R(u,h) \cdot h^r = o(h^{r+1})$. See Exercise 2.4-13 for the differentiability of R. The proof also shows that Taylor's formula holds if f is $(r-1)$ times differentiable on U and r times differentiable at u. The estimate $R(u,h) \cdot h^r = o(h^r)$ is proved directly by induction; for $r = 1$ it is the definition of the Fréchet derivative.

If f is C^∞ (i.e., is C^r for all r) then we may be able to extend Taylor's formula into a convergent power series. If we can, we say f is of class C^ω, or ***analytic***. A standard example of a C^∞ function that is not analytic is the following function from $\mathbb{R}$ to $\mathbb{R}$ (Figure 2.4.3)

$$\theta(x) = \begin{cases} \exp\left\{ -\frac{1}{1-x^2} \right\}, & |x| < 1; \\ 0, & |x| \ge 1. \end{cases}$$

This function is C^∞, and all derivatives are 0 at $x = \pm 1$. (To see this note that for $|x| < 1$,

$$f^{(n)}(x) = Q_n(x)(1-x^2)^{-2n} \exp\left(\frac{-1}{1-x^2} \right),$$

where $Q_n(x)$ are polynomials given recursively by

$$Q_0(x) = 1, \quad Q_{n+1}(x) = (1-x^2)^2 Q_n'(x) + 2x(2n-1-2nx^2) Q_n(x).)$$

Hence all coefficients of the Taylor series around these points vanish. Since the function is not identically 0 in any neighborhood of ± 1, it cannot be analytic there.

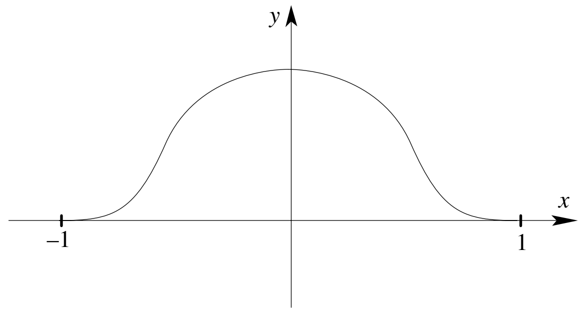

FIGURE 2.4.3. A bump function

2.4.16 Example (Differentiating Under the Integral)**.** Let $U \subset \mathbf{E}$ be open and $f : [a, b] \times U \to \mathbf{F}$. For $t \in [a, b]$, define $g(t) : U \to \mathbf{F}$ by $g(t)(u) = f(t, u)$. If, for each t, $g(t)$ is of class C^r and if the maps

$$(t, u) \in [a, b] \times U \mapsto \mathbf{D}^j(g(t))(u) \in L_s^j(\mathbf{E}, \mathbf{F})$$

are continuous, then $h : U \to \mathbf{F}$, defined by

$$h(u) = \int_a^b f(t, u)\, dt = \int_a^b g(t)(u)\, dt$$

is C^r and

$$\mathbf{D}^j h(u) = \int_a^b \mathbf{D}_u^j f(t, u)\, dt, \quad j = 1, \ldots, r,$$

where $\mathbf{D}_u$ means the partial derivative in u. For $r = 1$, write

$$\begin{aligned} &\left\| h(u + e) - h(u) - \int_a^b \mathbf{D}(g(t))(u) \cdot e\, dt \right\| \\ &= \left\| \int_a^b \left(\int_0^1 (\mathbf{D}(g(t))(u + se) \cdot e - \mathbf{D}(g(t))(u) \cdot e)\, ds \right) dt \right\| \\ &\leq (b - a)\|e\| \sup_{a \leq t \leq b, 0 \leq t \leq 1} \|\mathbf{D}(g(t))(u + se) - \mathbf{D}(g(t))(u)\| = o(e). \end{aligned}$$

For $r > 1$ one can also use an argument like this, but the converse to Taylor's theorem also yields the result rather easily. Indeed, if $R(t, u, e)$ denotes the remainder for the C^r Taylor expansion of $g(t)$, then with

$$\varphi_p = \mathbf{D}^p h = \int_a^b \mathbf{D}^p[g(t)]\, dt,$$

the remainder for h is clearly $R(u, e) = \int_a^b R(t, u, e)\, dt$. But $R(t, u, e)\, dt \to 0$ as $e \to 0$ uniformly in t, so $R(u, e)$ is continuous and $R(u, 0) = 0$. Thus h is C^r. ♦

SUPPLEMENT 2.4A
The Leibniz and Chain Rules

Here the explicit formulas are given for the kth order derivatives of products and compositions. The proofs are straightforward but quite messy induction arguments, which will be left to the interested reader.

The Higher Order Leibniz Rule. Let $\mathbf{E}$, $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{G}$ be Banach spaces, $U \subset \mathbf{E}$ an open set, $f : U \to \mathbf{F}_1$ and $g : U \to \mathbf{F}_2$ of class C^k and $B \in L(\mathbf{F}_1, \mathbf{F}_2; \mathbf{G})$. Let $f \times g : U \to \mathbf{F}_1 \times \mathbf{F}_2$ denote the mapping $(f \times g)(e) = (f(e), g(e))$ and let $B(f, g) = B \circ (f \times g)$. Thus $B(f, g)$ is of class C^k and by Leibniz' rule,

$$\mathbf{D}B(f,g)(p) \cdot e = B(\mathbf{D}f(p) \cdot e, g(p)) + B(f(p), \mathbf{D}g(p) \cdot e).$$

Higher derivatives of f and g are maps

$$\mathbf{D}^i f : U \to L^i(\mathbf{E}; \mathbf{F}_1), \quad \mathbf{D}^{k-i} g : U \to L^{k-i}(\mathbf{E}; \mathbf{F}_2),$$

where

$$\mathbf{D}^0 f = f, \quad \mathbf{D}^0 g = g, \quad L^0(\mathbf{E}; \mathbf{F}_1) = \mathbf{F}_1, \quad L^0(\mathbf{E}; \mathbf{F}_2) = \mathbf{F}_2.$$

Denote by

$$\lambda^{i,k-i} \in L(L^i(\mathbf{E}; \mathbf{F}_1), L^{k-i}(\mathbf{E}, \mathbf{F}_2); L^k(\mathbf{E}; \mathbf{G})),$$

the bilinear mapping defined by

$$[\lambda^{i,k-i}(A_1, A_2)](e_1, \ldots, e_k) = B(A_1(e_1, \ldots, e_i), A_2(e_{i+1}, \ldots, e_k))$$

for $A_1 \in L^i(\mathbf{E}; \mathbf{F}_1)$, $A_2 \in L^{k-i}(\mathbf{E}; \mathbf{F}_2)$, and $e_1, \ldots, e_k \in \mathbf{E}$. Then

$$\lambda^{i,k-i}(\mathbf{D}^i f, \mathbf{D}^{k-i} g) : U \to L^k(\mathbf{E}; \mathbf{G})$$

is defined by

$$\lambda^{i,k-i}(\mathbf{D}^i f, \mathbf{D}^{k-i} g)(p) = \lambda^{i,k-i}(\mathbf{D}^i f(p), \mathbf{D}^{k-i} g(p))$$

for $p \in U$. Leibniz' rule for kth derivatives is

$$\mathbf{D}^k B(f,g) = \operatorname{Sym}^k \circ \sum_{i=0}^{k} \binom{k}{i} \lambda^{i,k-i}(\mathbf{D}^i f, \mathbf{D}^{k-i} g),$$

where $\operatorname{Sym}^k : L^k(\mathbf{E}; \mathbf{G}) \to L^k_s(\mathbf{E}; \mathbf{G})$ is the symmetrization operator, given by (see Exercise 2.2-9):

$$(\operatorname{Sym}^k A)(e_1, \ldots, e_k) = \frac{1}{k!} \sum_{\sigma \in S_k} A(e_{\sigma(1)}, \ldots, e_{(k)}),$$

where S_k is the group of permutations of $\{1, \ldots, k\}$. Explicitly, taking advantage of the symmetry of higher order derivatives, this formula is

$$\begin{aligned} &\mathbf{D}^k B(f,g)(p) \cdot (e_1, \ldots, e_k) = \\ &\sum_{\sigma} \sum_{i=0}^{k} \binom{k}{i} B(\mathbf{D}^i f(p) \cdot (e_{\sigma(1)}, \ldots, e_{\sigma(i)}), \mathbf{D}^{k-i} g(p)(e_{\sigma(i+1)}, \ldots, e_{\sigma(k)})), \end{aligned} \tag{2.4.2}$$

where the outer sum is over all permutations $\sigma \in S_k$ such that

$$\sigma(1) < \cdots < \sigma(i) \quad \text{and} \quad \sigma(i+1) < \cdots < \sigma(k).$$

The Higher Order Chain Rule. Let $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$ be Banach spaces and $U \subset \mathbf{F}$ and $V \subset \mathbf{F}$ be open sets. Let $f : U \to V$ and $g : V \to \mathbf{G}$ be maps of class C^k. By the usual chain rule, $g \circ f : U \to \mathbf{G}$ is of class C^k and

$$\mathbf{D}(g \circ f)(p) = \mathbf{D}g(f(p)) \circ \mathbf{D}f(p)$$

for $p \in U$. For every tuple $(i, j_1, \ldots, j_i)$, where $i > 1$, and $j_1 + \cdots + j_i = k$, define the continuous multilinear map

$$\lambda^{i,j_1,\ldots,j_i} : L^i(\mathbf{F};\mathbf{G}) \times L^{j_1}(\mathbf{E};\mathbf{F}) \times \cdots \times L^{j_i}(\mathbf{E};\mathbf{F}) \to L^k(\mathbf{E};\mathbf{G})$$

by

$$\begin{aligned}\lambda^{i,j_1,\ldots,j_i}&(A, B_1, \ldots, B_i) \cdot (e_1, \ldots, e_k)\\ &= A(B_1(e_i, \ldots, e_{j_1}), \ldots, B_i(e_{j_i+\cdots+j_{1-1}+1}, \ldots, e_k))\end{aligned}$$

for

$$A \in L^i(\mathbf{F};\mathbf{G}), \quad B_\ell \in L^{j_\ell}(\mathbf{E};\mathbf{F}), \quad \ell = 1, \ldots, i \text{ and } e_i, \ldots, e_k \in \mathbf{E}.$$

Since $\mathbf{D}^{j_\ell} f : U \to L^{j_\ell}(\mathbf{E};\mathbf{F})$, we can define

$$\lambda^{i,j_1,\ldots,j_i} \circ (\mathbf{D}^i g \circ f \times \mathbf{D}^{j_1} f \times \cdots \times \mathbf{D}^{j_i}) : U \to L^k(\mathbf{E};\mathbf{G})$$

by

$$p \mapsto \lambda^{i,j_1,\ldots,j_i}(\mathbf{D}^i g(f(p)), \mathbf{D}^{j_1} f(p), \ldots, \mathbf{D}^{j_i} f(p)).$$

With these notations, the kth order chain rule is

$$\begin{aligned}\mathbf{D}^k(g \circ f) = \mathrm{Sym}^k \circ \sum_{i=1}^{k} \sum_{j_i+\cdots+j_1=k} &\frac{k!}{j_1! \cdots j_i!} \lambda^{i,j_1,\ldots,j_i}\\ &\circ (\mathbf{D}^i g \circ f \times \mathbf{D}^{j_1} f \times \cdots \times \mathbf{D}^{j_i} f),\end{aligned}$$

where $\mathrm{Sym}^k : L^k(\mathbf{E};\mathbf{G}) \to L^k_s(\mathbf{E};\mathbf{G})$ is the symmetrization operator. Taking into account the symmetry of higher order derivatives, the explicit formula at $p \in U$ and $e_1, \ldots, e_k \in \mathbf{E}$, is

$$\begin{aligned}&\mathbf{D}^k(g \circ f)(p) \cdot (e_1, \ldots, e_k)\\ &\quad = \sum_{i=1}^{k} \sum_{j_1+\cdots+j_i=k} \sum \mathbf{D}^i g(f(p))(\mathbf{D}^{j_1} f(p) \cdot (e_{\ell_1}, \ldots, e_{\ell_{j_1}}), \ldots,\\ &\qquad\qquad \mathbf{D}^{j_i} f(p) \cdot (e_{\ell_{j_1+\cdots+j_{i-1}+1}}, \ldots, e_{\ell_k}))\end{aligned}$$

where the third sum is taken for $\ell_1 < \cdots < \ell_{j_1} < \cdots < \ell_{j_1+\cdots+j_{i-1}+1} < \cdots < \ell_k$.

Supplement 2.4B

The Converse to Taylor's Theorem

This theorem goes back to Marcinkiewicz and Zygmund [1936], Whitney [1943a], and Glaeser [1958]. The proof of the converse that we shall follow is due to Nelson [1969]. Assume the formula in the theorem holds where $\varphi_p = \mathbf{D}^p f$, $1 \le p \le r$, and that $R(u,h)$ has the desired expression. If $r = 1$, the formula reduces to the definition of the derivative. Hence $\varphi_1 = \mathbf{D}f$, f is C^1, and thus $R(u,h)$ has the desired form, using Proposition 2.4.7. Inductively assume the theorem is true for $r = p-1$. Thus $\varphi_j = \mathbf{D}^j f$, for $1 \le j \le p-1$. Let $h, k \in \mathbf{E}$ be small in norm such that $u + h + k \in U$. Write the formula in the theorem for $f(u+h+k)$ in two different ways:

$$\begin{aligned} f(u+h+k) &= f(u+h) + \mathbf{D}f(u+h)\cdot k + \cdots \\ &\quad + \frac{1}{(p-1)!}\mathbf{D}^{p-1}f(u+h)\cdot k^{p-1} \\ &\quad + \frac{1}{p!}\varphi_p(u+h)\cdot k^p + R_1(u+k,k)\cdot k^p; \\ f(u+h+k) &= f(u) + \mathbf{D}f(u)\cdot(h+k) + \cdots \\ &\quad + \frac{1}{(p-1)!}\mathbf{D}^{p-1}f(u)\cdot(h+k)^{p-1} \\ &\quad + \frac{1}{p!}\varphi_p(u)\cdot(h+k)^p + R_2(u,h+k)\cdot(h+k)^p. \end{aligned}$$

Subtracting them and collecting terms homogeneous in k^j we get:

$$\begin{aligned} &g_0(h) + g_1(h)\cdot k + \cdots + g_{p-1}(h)\cdot k^{p-1} + g_p(h)\cdot k^p \\ &\quad = R_1(u+h,k)\cdot k^p - R_2(u,h+k)\cdot(h+k)^p, \end{aligned}$$

where $g_j(h) \in L^j(\mathbf{E};\mathbf{F})$, $g_j(0) = 0$ is given by

$$\begin{aligned} g_j(h) = \frac{1}{j!}\Bigg[&\mathbf{D}^j f(u+h) - \mathbf{D}^j f(u) - \sum_{i=1}^{p-1-j}\frac{1}{i!}\mathbf{D}^{j+1}f(u)\cdot h^i \\ &- \frac{1}{(p-j)!}\varphi_p(u)\cdot h^{p-j}\Bigg], \end{aligned}$$

where $0 \le j \le p-2$;

$$g_{p-1}(h) = \frac{1}{(p-1)!}\left[\mathbf{D}^{p-1}f(u+h) - \mathbf{D}^{p-1}f(u) - \varphi_p(u)\cdot h\right];$$

and

$$g_p(h) = \frac{1}{p!}\left[\varphi_p(u+h) - \varphi_p(u)\right].$$

Let $\|k\|$ satisfy $(1/4)\|h\| \le \|k\| \le (1/2)\|h\|$. Since

$$\begin{aligned} &\|R_1(u+h,k)\cdot k^p - R_2(u,h+k)\cdot(h+k)^p - g_p(h)\cdot k^p\| \\ &\quad \le (\|R_1(u+h,k)\| + \|g_p(h)\|)\,\|k\|^p + \|R_2(u,h+k)\|(\|h\| + \|k\|)^p \\ &\quad \le \{\|R_1(u+h,k)\| + \|g_p(h)\| + \|R_2(u,h+k)\|\}\,(1+3^p)\|h\|^p/2^p \end{aligned}$$

and the quantity in braces $\{\} \to 0$ as $h \to 0$, it follows that

$$R_1(u+h,k)\cdot k^p - R_2(u,h+k)\cdot(h+k)^p - g_p(h)\cdot k^p = o(h^p).$$

Hence

$$g_0(h) + g_1(h)\cdot k + \cdots + g_{k-1}(h)\cdot k^{p-1} = o(h^p).$$

We claim that subject to the condition $(1/4)\|h\| \le \|k\| \le (1/2)\|h\|$, each term of this sum is $o(h^p)$. If $\lambda_1, \ldots, \lambda_p$ are *distinct* numbers, replace k by $\lambda_j k$ in the foregoing, and get a $p \times p$ linear system in the unknowns $g_0(h), \ldots, g_{p-1}(h) \cdot k^{p-1}$ with Vandermonde determinant $\Pi_{i<j}(\lambda_i - \lambda_j) \neq 0$ and right-hand side a column vector all of whose entries are $o(h^p)$. Solving this system we get the result claimed. In particular,

$$(\mathbf{D}^{p-1} f(u+k) - \mathbf{D}^{p-1} f(u) - \varphi_p(u) \cdot h) \cdot k^{p-1} = g_{p-1}(h) \cdot k^{p-1} = o(h^p).$$

Using polarization (see Supplement 2.2B) we get

$$\begin{aligned}
&\|\mathbf{D}^{p-1} f(u+h) - \mathbf{D}^{p-1} f(u) - \varphi_p(u) \cdot h\| \\
&\quad \le \frac{(p-1)^{p-1}}{(p-1)!} \left\| (\mathbf{D}^{p-1} f(u+h) - \mathbf{D}^{p-1} f(u) - \varphi_p(u) \cdot h)' \right\| \\
&\quad = \frac{(p-1)^{p-1}}{(p-1)!} \sup_{\|e\| \le 1} \left\| (\mathbf{D}^{p-1} f(u+h) - \mathbf{D}^{p-1} f(u) \right. \\
&\qquad\qquad\qquad\qquad \left. - \varphi_p(u) \cdot h) \cdot e^{p-1} \right\| \\
&\quad = \frac{(p-1)^{p-1}}{(p-1)!} \sup_{\|k\| \le \|h\|/2} \left\| (\mathbf{D}^{p-1} f(u+h) - \mathbf{D}^{p-1} f(u) \right. \\
&\qquad\qquad\qquad\qquad \left. - \varphi_p(u) \cdot h) \cdot \left(\frac{2k}{\|h\|} \right)^{p-1} \right\| \\
&\quad = \frac{(2(p-1))^{p-1}}{(p-1)!\|h\|^{p-1}} \sup_{\|k\| \le \|h\|/2} \left\| (\mathbf{D}^{p-1} f(u+h) - \mathbf{D}^{p-1} f(u) \right. \\
&\qquad\qquad\qquad\qquad \left. - \varphi_p(u) \cdot h) \cdot k^{p-1} \right\| \\
&\quad = \frac{(2(p-1))^{p-1}}{(p-1)!\|h\|^{p-1}} o(h^p)
\end{aligned}$$

Since $o(h^p)/\|h\|^p \to 0$ as $h \to 0$, this relation proves that $\mathbf{D}^{p-1} f$ is differentiable and $\mathbf{D}^p f(u) = \varphi_p(u)$. Thus f is of class C^p, φ_p being continuous, and the formula for R follows by subtracting the given formula for $f(u+h)$ from Taylor's expansion.

The converse of Taylor's theorem provides an alternative proof that $\mathbf{D}^r f(u) \in L_s^r(\mathbf{E}; \mathbf{F})$. Observe first that in the proof of Taylor's expansion for a C^r map f the symmetry of $\mathbf{D}^j f(u)$ was never used, so if one symmetrizes the $\mathbf{D}^j f(u)$ and calls them φ_j, the same expansion holds. But then the converse of Proposition 2.4.12 says that $\varphi_j = \mathbf{D}^j f$.

We shall consider here simple versions of two theorems from global analysis, which shall be used in Supplement 4.1C, namely the smoothness of the evaluation mapping and the "omega lemma."

The Evaluation Map. Let $I = [0, 1]$ and $\mathbf{E}$ be a Banach space. The vector space $C^r(I; \mathbf{E})$ of C^r-maps $(r > 0)$ of I into $\mathbf{E}$ is a Banach space with respect to the norm

$$\|f\|_k = \max_{1 \le i \le k} \sup_{t \in I} \|\mathbf{D}^i f(t)\|$$

(see Exercise 2.4-8). If U is open in $\mathbf{E}$, then the set

$$C^r(I; U) = \{ f \in C^r(I; \mathbf{E}) \mid f(I) \subset U \}$$

is checked to be open in $C^r(I; \mathbf{E})$.

2.4.17 Proposition. *The evaluation map defined by:*

$$\mathrm{ev} : C^r(I; U) \times \,]0,1[\, \to U$$

defined by

$$\mathrm{ev}(f, t) = f(t)$$

is C^r and its kth derivative is given by

$$\begin{aligned}\mathbf{D}^k \,\mathrm{ev}(f,t) \cdot ((g_1, s_1), \ldots, (g_k, s_k))\\ = \mathbf{D}^k f(t) \cdot (s_1, \ldots, s_k) + \sum_{i=1}^{k} \mathbf{D}^{i-1} g_i(t) \cdot (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_k)\end{aligned}$$

where

$$(g_i, s_i) \in C^r(I; \mathbf{E}) \times \mathbb{R}, \quad i = 1, \ldots, k.$$

Proof. For $(g, s) \in C^r(I; \mathbf{E}) \times \mathbb{R}$, define the norm $\|(g, s)\| = \max(\|g\|_k, |s|)$. Note that the right-hand side of the formula in the statement is symmetric in the arguments (g_i, s_i), $i = 1, \ldots, k$. We shall let this right-hand side be denoted

$$\varphi_k : C^r(I; U) \times \,]0,1[\, \to L^k_s(C^r(I; \mathbf{E}) \times \mathbb{R}; \mathbf{E}).$$

Note that $\varphi_0(f, t) = f(t)$ and that the proposition holds for $r = 0$ by uniform continuity of f on I since

$$\|f(t) - g(s)\| \leq \|f(t) - f(s)\| + \|f - g\|_0.$$

Since

$$\lim_{(g,s)\to(0,0)} \frac{\mathbf{D}^r g(t) \cdot s^r}{\|(g, s)\|^r} = 0$$

for all $t \in \,]0,1[$, by Taylor's theorem for g we get

$$\begin{aligned}\mathrm{ev}(f + g, t + s) &= f(t + s) + g(t + s)\\ &= \sum_{i=0}^{r} \frac{1}{i!}(\mathbf{D}^i f(t) \cdot s^i + \mathbf{D}^i g(t) \cdot s^i) + R(t, s) \cdot s^r\\ &= f(t) + \sum_{i=0}^{r} \frac{1}{i!}\varphi_i(f, t) \cdot (g, s)^i + R((f, t), (g, s)) \cdot (g, s)^r,\end{aligned}$$

where

$$\begin{aligned}R((f,t),(g,s)) \cdot ((g_1, s_1), \ldots, (g_r, s_r)) = R(t, s) \cdot (s_1, \ldots, s_r)\\ + \sum_{i=1}^{r} \mathbf{D}^r g_i(t) \cdot (s_1, \ldots, s_r),\end{aligned}$$

which is symmetric in its arguments and $R((f, t), (0, 0)) = 0$. By the converse to Taylor's theorem, the proposition is proved if we show that every φ_i, $1 \leq i \leq r$, is continuous. Since

$$\|\mathbf{D}^{k-1} g_i(t) - \mathbf{D}^{k-1} g_i(s)\| \leq |t - s| \sup_{u \in I} \|\mathbf{D}^k g_i(u)\| \leq |t - s| \, \|g_i\|_r$$

by the mean value theorem, the inequality

$$\begin{aligned}\|(\varphi_k(f,t) - \varphi_k(g,s)) \cdot ((g_1,s_1),\ldots,(g_k,s_k))\| \\ \le \|\mathbf{D}^k f(t) - \mathbf{D}^k g(s)\|\, |s_1|\cdots|s_k| \\ + \sum_{i=1}^{k} \|\mathbf{D}^{k-1} g_i(t) - \mathbf{D}^{k-1} g_i(s)\|\, |s_1|\cdots|s_{i-1}|\,|s_{i+1}|\cdots|s_k|\end{aligned}$$

implies

$$\begin{aligned}\|\varphi_k(f,t) - \varphi_k(g,s)\| &\le \|\mathbf{D}^k f(t) - \mathbf{D}^k g(s)\| + k|t-s| \\ &\le \|\mathbf{D}^k f(t) - \mathbf{D}^k f(s)\| + \|\mathbf{D}^k f(s) - \mathbf{D}^k g(s)\| \\ &\quad + k|t-s| \\ &\le \|\mathbf{D}^k f(t) - \mathbf{D}^k f(s)\| + 2k\|(f,t) - (g,s)\|.\end{aligned}$$

Thus the uniform continuity of $\mathbf{D}^k f$ on I implies the continuity of φ_k at (f,t). ■

Omega Lemma. (This is terminology of Abraham [1963]. Various results of this type can be traced back to earlier works of Sobolev [1939] and Eells [1958].)

Let M be a compact topological space and $\mathbf{E}, \mathbf{F}$ be Banach spaces. With respect to the norm

$$\|f\| = \sup_{m \in M} \|f(m)\|,$$

the vector space $C^0(M,\mathbf{E})$ of continuous $\mathbf{E}$-valued maps on M, is a Banach space. If U is open in $\mathbf{E}$, it is easy to see that

$$C^0(M,U) = \{\, f \in C^0(M,\mathbf{E}) \mid f(M) \subset U \,\}$$

is open.

2.4.18 Lemma (Omega Lemma). *Let $g : U \to \mathbf{F}$ be a C^r map, $r > 0$. The map*

$$\Omega_g : C^0(M,U) \to C^0(M,\mathbf{F}) \quad \text{defined by } \Omega_g(f) = g \circ f$$

is also of class C^r. The derivative of Ω_g is

$$\mathbf{D}\Omega_g(f) \cdot h = [(\mathbf{D}g) \circ f] \cdot h$$

that is,

$$[\mathbf{D}\Omega_g(f) \cdot h](x) = \mathbf{D}g(f(x)) \cdot h(x).$$

The formula for $\mathbf{D}\Omega_g$ is quite plausible. Indeed, we have

$$[\mathbf{D}\Omega_g(f) \cdot h](x) = \left.\frac{d}{d\varepsilon}\Omega_g(f + \varepsilon h)(x)\right|_{\varepsilon=0} = \left.\frac{d}{d\varepsilon} g(f(x) = \varepsilon h(x))\right|_{\varepsilon=0}$$

By the chain rule this is $\mathbf{D}g(f(x)) \cdot h(x)$. This shows that if Ω_g is differentiable, then $\mathbf{D}\Omega_g$ must be as stated in the proposition.

Proof. Let $f \in C^0(M,U)$. By continuity of g and compactness of M,

$$\|\Omega_g(f) - \Omega_g(f')\| = \sup_{m \in M} \|g(f(m)) - g(f'(m))\|$$

is small as soon as $\|f - f'\|$ is small; that is, Ω_g is continuous at each point f. Let

$$A_i : C^0(M, L^i_s(\mathbf{E}; \mathbf{F})) \to L^i_s(C^0(M, \mathbf{E}); C^0(M, \mathbf{F}))$$

be given by

$$A_i(H)(h_1, \ldots, h_i)(m) = H(m)(h_1(m), \ldots, h_i(m))$$

for $H \in C^0(M, L^i(\mathbf{E}; \mathbf{F}))$, $h_1, \ldots, h_i \in C^0(M, \mathbf{E})$ and $m \in M$. The maps A^i are clearly linear and are continuous with $\|A^i\| \leq 1$. Since $\mathbf{D}^i g : U \to L^i_s(\mathbf{E}; \mathbf{F})$ is continuous, the preceding argument shows that the maps

$$\Omega_{\mathbf{D}^i g} : C^0(M, U) \to L^i_s(C^0(M, \mathbf{E}); C^0(M, \mathbf{F}))$$

are continuous and hence

$$A_i \circ \Omega_{\mathbf{D}^i g} : C^0(M, U) \to L^i_s(C^0(M, \mathbf{E}); C^0(M, \mathbf{F}))$$

is continuous. The Taylor theorem applied to g yields

$$\begin{aligned} g(f(m) + h(m)) = g(f(m)) &+ \sum_{i=1}^{r} \frac{1}{i!} \mathbf{D}^i g(f(m)) \cdot h(m)^i \\ &+ R(f(m), h(m)) \cdot h(m)^i \end{aligned}$$

so that defining

$$[(\mathbf{D}^i g \circ f) \cdot h^i](m) = \mathbf{D}^i g(f(m)) \cdot h(m)^i,$$

and

$$[R(f, h) \cdot (h_1, \ldots, h_r)](m) = R(f(m), h(m)) \cdot (h_1(m), \ldots, h_r(m))$$

we see that R is continuous, $R(f, 0) = 0$, and

$$\begin{aligned} \Omega_g(f + h) = g \circ (f + h) &= g \circ f + \sum_{i=1}^{r} \frac{1}{i!} (\mathbf{D}^i g \circ f) \cdot h^i + R(f, h) \cdot h^i \\ &= \Omega_g(f) + \sum_{i=1}^{r} \frac{1}{i!} (A_i \circ \Omega_{\mathrm{D}^i g})(f) \cdot h^i + R(f, h) \cdot h^i. \end{aligned}$$

Thus by the converse of Taylor's theorem, $\mathbf{D}^i \Omega_g = A_i \circ \Omega_{\mathrm{D}^i g}$ and Ω_g is of class C^r. ■

This proposition can be generalized to the Banach space $C^r(I, \mathbf{E})$, $I = [a, b]$, equipped with the norm $\| \cdot \|_r$ given by the maximum of the norms of the first r derivatives; that is,

$$\|f\|_r = \max_{0 \leq i \leq r} \sup_{t \in I} \|f^{(i)}(t)\|.$$

If g is C^{r+q}, then $\Omega_g : C^r(I, \mathbf{E}) \to C^{r-k}(I, \mathbf{F})$ is C^{q+k}. Readers are invited to convince themselves that the foregoing proof works with only trivial modifications in this case. This version of the omega lemma will be used in Supplement 4.1C.

For applications to partial differential equations, the most important generalizations of the two previous propositions is to the case of Sobolev maps of class H^s; see for example Palais [1968], Ebin and Marsden [1970], and Marsden and Hughes [1983] for proofs and applications.

Supplement 2.4C
The Functional Derivative and the Calculus of Variations

Differential calculus in infinite dimensions has many applications, one of which is to the calculus of variations. We give some of the elementary aspects here. We shall begin with some notation and a generalization of the notion of the dual space.

Duality and Pairings. Let $\mathbf{E}$ and $\mathbf{F}$ be Banach spaces. A continuous bilinear functional $\langle\,,\rangle : \mathbf{E}\times\mathbf{F}\to\mathbb{R}$ is called $\mathbf{E}$-***non-degenerate*** if $\langle x,y\rangle = 0$ for all $y\in\mathbf{F}$ implies $x=0$. Similarly, it is $\mathbf{F}$-***non-degenerate*** if $\langle x,y\rangle = 0$ for all $x\in\mathbf{E}$ implies $y=0$. If it is both, we just say $\langle\,,\rangle$ is ***non-degenerate***. Equivalently, the two linear maps of $\mathbf{E}$ to $\mathbf{F}^*$ and $\mathbf{F}$ to $\mathbf{E}^*$ defined by $x\mapsto\langle x,\cdot\rangle$ and $y\mapsto\langle\cdot,y\rangle$, respectively, are one to one. If they are isomorphisms, $\langle\,,\rangle$ is called $\mathbf{E}$- or $\mathbf{F}$-***strongly non-degenerate***. A non-degenerate bilinear form $\langle\,,\rangle$ thus represents certain linear functionals on $\mathbf{F}$ in terms of elements in $\mathbf{E}$. We say $\mathbf{E}$ and $\mathbf{F}$ are in ***duality*** if there is a non-degenerate bilinear functional $\langle\,,\rangle : \mathbf{E}\times\mathbf{F}\to\mathbb{R}$, also called a ***pairing*** of $\mathbf{E}$ with $\mathbf{F}$. If the functional is strongly non-degenerate, we say the duality is ***strong***.

2.4.19 Examples.

A. Let $\mathbf{E}=\mathbf{F}^*$. Let $\langle\,,\rangle : \mathbf{F}^*\times\mathbf{F}\to R$ be given by $\langle\varphi,y\rangle=\varphi(y)$ so the map $\mathbf{E}\to\mathbf{F}^*$ is the identity. Thus, $\langle\,,\rangle$ is $\mathbf{E}$-strongly non-degenerate by the Hahn–Banach theorem. It is easily checked that $\langle\,,\rangle$ is $\mathbf{F}$-non-degenerate. (If it is $\mathbf{F}^*$ strongly non-degenerate, $\mathbf{F}$ is called ***reflexive***.)

B. Let $\mathbf{E}=\mathbf{F}$ and $\langle\,,\rangle : \mathbf{E}\times\mathbf{E}\to\mathbb{R}$ be an inner product on $\mathbf{E}$. Then $\langle\,,\rangle$ is non-degenerate since $\langle\,,\rangle$ is positive definite. If $\mathbf{E}$ is a Hilbert space, then $\langle\,,\rangle$ is a strongly non-degenerate pairing by the Riesz representation theorem. ♦

Functional Derivatives. We now define the functional derivative which uses the pairing similar to how one defines the gradient.

2.4.20 Definition. *Let $\mathbf{E}$ and $\mathbf{F}$ be normed spaces and $\langle\,,\rangle$ be an $\mathbf{E}$-weakly non-degenerate pairing. Let $f:\mathbf{F}\to\mathbb{R}$ be differentiable at the point $\alpha\in\mathbf{F}$. The **functional derivative** $df/d\alpha$ of f with respect to α is the unique element in $\mathbf{E}$, if it exists, such that*

$$\mathbf{D}f(\alpha)\cdot\beta = \left\langle \frac{\delta f}{\delta\alpha},\beta\right\rangle \quad \text{for all } \beta\in\mathbf{F}. \tag{2.4.3}$$

*Likewise, if $g:\mathbf{E}\to\mathbb{R}$ and $\langle\,,\rangle$ is $\mathbf{F}$-weakly degenerate, we define the **functional derivative** $\delta g/\delta v\in\mathbf{F}$, if it exists, by*

$$\mathbf{D}g(v)\cdot v' = \left\langle v',\frac{\delta f}{\delta v}\right\rangle \quad \text{for all } v'\in\mathbf{E}$$

Often $\mathbf{E}$ and $\mathbf{F}$ are spaces of mappings, as in the following example.

2.4.21 Example. Let $\Omega\in\mathbb{R}^n$ be an open bounded set and consider the space $\mathbf{E}=C^0(D)$, of continuous real valued functions on D where $D=\mathrm{cl}(\Omega)$. Take $\mathbf{F}=C^0(D)=\mathbf{E}$. The L^2-pairing on $\mathbf{E}\times\mathbf{F}$ is the bilinear map given by

$$\langle\,,\rangle : C^0(D)\times C^0(D)\to\mathbb{R},\quad \langle f,g\rangle=\int_\Omega f(x)g(x)\,d^nx.$$

Let r be a positive integer and define $f:\mathbf{E}\to\mathbb{R}$ by

$$f(\varphi)=\frac{1}{2}\int_\Omega[\varphi(x)]^r\,d^nx.$$

Then using the calculus rules from this section, we find

$$\mathbf{D}f(\varphi)\cdot\psi = \int_\Omega r[\varphi(r)]^{r-1}\psi(s)\,d^nx.$$

Thus, $\dfrac{\delta f}{\delta\varphi} = r\varphi^{r-1}$. ♦

Suppose, more generally, that f is defined on a Banach space $\mathbf{E}$ of functions φ on a region Ω in $\mathbb{R}^n$. The functional derivative $(\delta f/\delta\varphi)$ of f with respect to φ is the unique element $(\delta f/\delta\varphi) \in \mathbf{E}$, if it exists, such that

$$\mathbf{D}f(\varphi)\cdot\psi = \left\langle \frac{\delta f}{\delta\varphi}, \psi\right\rangle = \int_\Omega \left(\frac{\delta f}{\delta\varphi}\right)(x)\psi(x)\,d^nx \quad \text{for all } \psi \in \mathbf{E}.$$

The functional derivative may be determined in examples by

$$\int_\Omega \frac{\delta f}{\delta\varphi}(x)\psi(x)\,d^nx = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} f(\varphi+\varepsilon\psi). \tag{2.4.4}$$

Criterion for Extrema. A basic result in the calculus of variations is the following.

2.4.22 Proposition. *Let $\mathbf{E}$ be a space of functions, as above. A necessary condition for a differentiable function $f : \mathbf{E} \to \mathbb{R}$ to have an extremum at φ is that*

$$\frac{\delta f}{\delta\varphi} = 0.$$

Proof. If f has an extremum at φ, then for each ψ, the function $h(\varepsilon) = f(\varphi+\varepsilon\psi)$ has an extremum at $\varepsilon = 0$. Thus, by elementary calculus, $h'(0) = 0$. Since ψ is arbitrary, the result follows from equation (2.4.4). ■

Sufficient conditions for extrema in the calculus of variations are more delicate. See, for example, Bolza [1904] and Morrey [1966].

2.4.23 Examples.

A. Suppose that $\Omega \subset \mathbb{R}$ is an interval and that f, as a functional of $\varphi \in C^k(\Omega)$, $k \geq 1$, is of the form

$$f(\varphi) = \int_\Omega F\left(x, \varphi(x), \frac{d\varphi}{dx}\right)dx \tag{2.4.5}$$

for some smooth function $F : \Omega\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, so that the right hand side of equation (2.4.5) is defined. We call F the ***density*** associated with f. It can be shown by using the results of the preceding supplement that f is smooth. Using the chain rule,

$$\begin{aligned}\int_\Omega \frac{\delta f}{\delta\varphi}(x)\psi(x)\,dx &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\int_\Omega F\left(x, \varphi+\varepsilon\psi, \frac{d(\varphi+\varepsilon\psi)}{dx}\right)dx\\ &= \int_\Omega \mathbf{D}_2F\left(x,\varphi(x),\frac{d\varphi}{dx}\right)\psi(x)\,dx\\ &\quad+\int_\Omega \mathbf{D}_3F\left(x,\varphi(x),\frac{d\varphi}{dx}\right)\frac{d\psi}{dx}\,dx,\end{aligned}$$

where

$$\mathbf{D}_2F = \frac{\partial f}{\partial\varphi} \quad\text{and}\quad \mathbf{D}_3F = \frac{\partial F}{\partial(\partial\varphi/\partial x)}$$

denote the partial derivatives of F with respect to its second and third arguments. Integrating by parts, this becomes

$$\int_\Omega \mathbf{D}_2 F\left(x, \varphi(x), \frac{d\varphi}{dx}\right)\psi(x)\,dx - \int_\Omega \left(\frac{d}{dx}\mathbf{D}_3 F\left(x, \varphi(x), \frac{d\varphi}{dx}\right)\right)\psi(x)\,dx \\ + \int_{\partial\Omega} \mathbf{D}_3 F\left(x, \varphi(x), \frac{d\varphi}{dx}\right)\psi(x)\,dx.$$

Let us now restrict our attention to the space of ψ's which vanish on the boundary $\partial\Omega$ of Ω . In that case we get

$$\frac{\delta f}{\delta\varphi} = \mathbf{D}_2 F - \frac{d}{dx}\mathbf{D}_3 F.$$

Rewriting this according to the designation of the second and third arguments of F as φ and $d\varphi/dx$, respectively, we obtain

$$\frac{\delta f}{\delta\varphi} = \frac{\partial F}{\partial\varphi} - \frac{d}{dx}\frac{\partial F}{\partial(d\varphi/dx)}. \tag{2.4.6}$$

By a similar argument, if $\Omega \subset \mathbb{R}^n$, equation (2.4.6) generalizes to

$$\frac{\delta f}{\delta\varphi} = \frac{\partial F}{\partial\varphi} - \frac{d}{dx^k}\frac{\partial F}{\partial(d\varphi/dx^k)}. \tag{2.4.7}$$

(Here, a sum on repeated indices is assumed.) Thus, f has an extremum at φ only if

$$\frac{\partial F}{\partial\varphi} - \frac{d}{dx^k}\frac{\partial F}{\partial(\partial\varphi/\partial x^k)} = 0.$$

This is called the ***Euler–Lagrange*** equation in the calculus of variations.

B. Assume that in Example A, the density F associated with f depends also on higher derivatives, that is, $F = F(x, \varphi(x), \varphi_x, \varphi_{xx}, \ldots)$, where $\varphi_x = d\varphi/dx$, $\varphi_{xx} = d^2\varphi/dx^2$, etc. Therefore

$$f(\varphi) = \int_\Omega F(x, \varphi(x), \varphi_x, \varphi_{xx}, \ldots)\,dx.$$

By an analogous argument, formula (2.4.6) generalizes to

$$\frac{\delta f}{\delta\varphi} = \frac{\partial F}{\partial\varphi} - \frac{d}{dx}\left(\frac{\partial F}{\partial\varphi_x}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial\varphi_{xx}}\right) - \cdots \tag{2.4.8}$$

C. Consider a closed curve γ in $\mathbb{R}^3$ such that γ lies above the boundary $\partial\Omega$ of a region Ω in the xy-plane, as in Figure 2.4.4.

Consider differentiable surfaces in $\mathbb{R}^3$ (i.e., two-dimensional manifolds of $\mathbb{R}^3$) that are graphs of C^k functions $\varphi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, so that $(x, y, \varphi(x, y))$ are coordinates on the surface. What is the surface of least area whose boundary is γ? From elementary calculus we know that the area as a function of φ is given by

$$A(\varphi) = \int_\Omega \sqrt{1 + \varphi_x^2 + \varphi_y^2}\,dx\,dy.$$

From equation (2.4.7), a necessary condition for φ to minimize A is that

$$\frac{\delta A}{\delta\varphi} = -\frac{\varphi_{xx}(1+\varphi_y^2) - 2\varphi_x\varphi_y\varphi_{xy} + \varphi_{yy}(1+\varphi_x^2)}{(1+\varphi_x^2+\varphi_y^2)^{exc:3.2-27}} = 0, \tag{2.4.9}$$

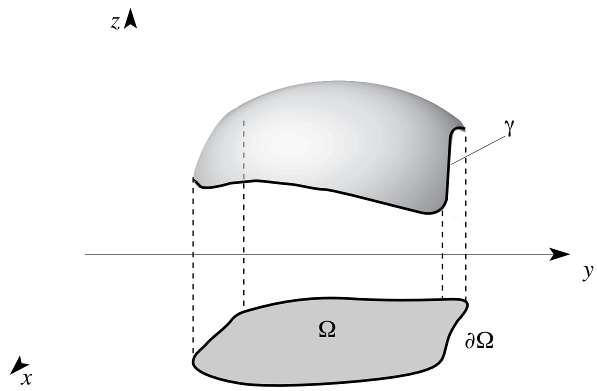

FIGURE 2.4.4. A curve γ lying over $\partial\Omega$

for $(x, y) \in \Omega$. We relate this to the classical theory of surfaces as follows. A surface has two principal curvatures κ_1 and κ_2; the ***mean curvature*** κ is defined to be their average: that is, $\kappa = (\kappa_1 + \kappa_2)/2$. An elementary theorem of geometry asserts that κ is given by the formula

$$\kappa = \frac{\varphi_{xx}(1+\varphi_y^2) - 2\varphi_x\varphi_y\varphi_{xy} + \varphi_{yy}(1+\varphi_x^2)}{(1+\varphi_x^2+\varphi_y^2)^{1/2}}. \tag{2.4.10}$$

If the surface represents a sheet of rubber, the mean curvature represents the net force due to internal stretching. Comparing equations (2.4.9) and (2.4.10) we find the well-known result that a minimal surface, that is, a surface with minimal area, has zero mean curvature. ♦

Total Functional Derivative. Now consider the case in which f is a differentiable function of n variables, that is f is defined on a product of n function spaces $\mathbf{F}_i$, $i = 1, \ldots, n$; $f : \mathbf{F}_1 \times \cdots \times \mathbf{F}_n \to \mathbb{R}$ and we have pairings $\langle\,,\rangle_i : \mathbf{E}_i \times \mathbf{F}_i \to \mathbb{R}$.

2.4.24 Definition. *The i-th partial functional derivative $\delta f/\delta\varphi_i$ of f with respect to $\varphi_i \in \mathbf{F}_i$ is defined by*

$$\begin{aligned}\left\langle \frac{\delta f}{\delta\varphi_i}, \psi_i \right\rangle_i &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} f(\varphi_1, \ldots, \varphi_i + \varepsilon\psi_i, \ldots, \varphi_n) \\ &= \mathbf{D}_i f(\varphi_1, \ldots, \varphi_n) \cdot \psi_i = \mathbf{D} f(\varphi_1, \ldots, \varphi_n)(0, \ldots, \psi_i, \ldots, 0).\end{aligned} \tag{2.4.11}$$

*The **total functional derivative** is given by*

$$\begin{aligned}\left\langle \frac{\delta f}{\delta(\varphi_1, \ldots, \varphi_n)}, (\psi_1, \ldots, \psi_n) \right\rangle &= \mathbf{D} f(\varphi_1, \ldots, \varphi_n) \cdot (\psi_1, \ldots, \psi_n) \\ &= \sum_{i=1}^n \mathbf{D}_i f(\varphi_1, \ldots, \varphi_n)(0, \ldots, \psi_i, \ldots, 0) \\ &= \sum_{i=1}^n \left\langle \frac{\delta f}{\delta\varphi_i}, \psi_i \right\rangle_i .\end{aligned}$$

2.4.25 Examples.

A. Suppose that f is a function of n functions $\varphi_i \in C^k(\Omega)$, where $\Omega \subset \mathbb{R}^n$, and their first partial derivatives, and is of the form

$$f(\varphi_1, \ldots, \varphi_n) = \int_\Omega F\left(x, \varphi_i, \frac{\partial\varphi_i}{\partial x^i}\right) d^n x.$$

It follows that

$$\frac{\delta f}{\delta \varphi_i} = \frac{\partial F}{\partial \varphi_i} - \frac{\partial}{\partial x^k} \frac{\partial F}{\partial \left(\dfrac{\partial \varphi_i}{\partial x^k} \right)} \quad \text{(sum on } k\text{)} \tag{2.4.12}$$

B. Classical Field Theory. As discussed in Goldstein [1980, Section 12], Lagrange's equations for a field $\eta = \eta(x,t)$ with components η^a follow from Hamilton's variational principle. When the Lagrangian L is given by a Lagrangian density $\pounds$, that is, L is of the form

$$L(\eta) = \iint_{\Omega \subset \mathbb{R}^3} \pounds \left(x^j, \eta^a, \frac{\partial \eta^a}{\partial x^j}, \frac{\partial \eta^a}{\partial t} \right) d^n x \, dt \tag{2.4.13}$$

the variational principle states that η should be a critical point of L. Assuming appropriate boundary conditions, this results in the ***equations of motion***

$$0 = \frac{\delta L}{\delta \eta^a} = \frac{d}{dt} \frac{\partial \pounds}{\partial (\partial \eta^a / \partial t)} - \frac{\partial \pounds}{\partial \eta^a} + \frac{\partial}{\partial x^k} \frac{\partial \pounds}{\partial (\partial \eta^a / \partial x^k)} \tag{2.4.14}$$

(sum on k is understood). Regarding L as a function of η^a and $\dot{\eta^a} = \partial \eta^a / \partial t$, the equations of motion take the form:

$$\frac{d}{dt} \frac{\delta L}{\delta \dot{\eta^a}} = \frac{\delta L}{\delta \eta^a} \tag{2.4.15}$$

C. Let $\Omega \subset \mathbb{R}^n$ and let $C^k_\partial(\Omega)$ stand for the C^k functions vanishing on $\partial\Omega$. Let $f : C^k_\partial(\Omega) \to \mathbb{R}$ be given by the Dirichlet integral

$$f(\varphi) = \frac{1}{2} \int_\Omega \langle \nabla \varphi, \nabla \varphi \rangle \, d^n x.$$

Using the standard inner product $\langle \, , \rangle$ in $\mathbb{R}^n$, we may write

$$f(\varphi) = \frac{1}{2} \int_\Omega \langle \nabla \varphi, \nabla \varphi \rangle \; d^n x.$$

Differentiating with respect to φ:

$$\begin{aligned} \mathbf{D} f(\varphi) \cdot \psi &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \frac{1}{2} \int_\Omega \langle \nabla (\varphi + \varepsilon \psi), \nabla (\varphi + \varepsilon \psi) \rangle \; d^n x \\ &= \int_\Omega \langle \nabla \varphi, \nabla \psi \rangle \; d^n x \\ &= - \int_\Omega \nabla^2 \varphi(x) \cdot \psi(x) \, d^n x \qquad \text{(integrating by parts).} \end{aligned}$$

Thus $\delta f / \delta \varphi = -\nabla^2 \varphi$, the Laplacian of φ.

D. The Stretched String. Consider a string of length ℓ and mass density σ, stretched horizontally under a tension τ, with ends fastened at $x = 0$ and $x = \ell$. Let $u(x,t)$ denote the vertical displacement of the string at x, at time t. We have $u(0,t) = u(\ell,t) = 0$. The potential energy V due to small vertical displacements is shown in elementary mechanics texts to be

$$V = \int_0^\ell \frac{1}{2} \tau \left(\frac{\partial u}{\partial x} \right)^2 dx,$$

and the kinetic energy T is

$$T = \int_0^\ell \frac{1}{2}\sigma \left(\frac{\partial u}{\partial t}\right)^2 dx.$$

From the definitions, we get

$$\frac{\delta V}{\delta u} = -\tau \frac{\partial^2 u}{\partial x^2} \quad \text{and} \quad \frac{\delta T}{\delta \dot{u}} = \sigma \dot{u}.$$

Then with the Lagrangian $L = T - V$, the equations of motion (2.4.15) become the ***wave equation***

$$\sigma \frac{\partial^2 u}{\partial t^2} - \tau \frac{\partial^2 u}{\partial x^2} = 0. \qquad \blacklozenge$$

Next we formulate a chain rule for functional derivatives. Let $\langle \, , \rangle : \mathbf{E} \times \mathbf{F} \to \mathbb{R}$ be a weakly nondegenerate pairing between $\mathbf{E}$ and $\mathbf{F}$. If $A \in L(\mathbf{F}, \mathbf{F})$, its ***adjoint*** $A^* \in L(\mathbf{E}, \mathbf{E})$, if it exists, is defined by $\langle A^* v, \alpha \rangle = \langle v, A\alpha \rangle$ for all $v \in \mathbf{E}$ and $\alpha \in \mathbf{F}$.

Let $\varphi : \mathbf{F} \to \mathbf{F}$ be a differentiable map and $f : \mathbf{F} \to \mathbb{R}$ be differentiable at $\alpha \in \mathbf{F}$. From the chain rule,

$$\mathbf{D}(f \circ \varphi)(\alpha) \cdot \beta = \mathbf{D} f(\varphi(\alpha)) \cdot (\mathbf{D}\varphi(\alpha) \cdot \beta), \quad \text{for } \beta \in \mathbf{F}.$$

Hence assuming that all functional derivatives and adjoints exist, the preceding relation implies

$$\left\langle \frac{\delta (f \circ \varphi)}{\delta \alpha}, \beta \right\rangle = \left\langle \frac{\delta f}{\delta \gamma}, \mathbf{D}\varphi(\alpha) \cdot \beta \right\rangle = \left\langle \mathbf{D}\varphi(\alpha)^* \cdot \frac{\delta f}{\delta \gamma}, \beta \right\rangle$$

where $\gamma = \varphi(\alpha)$, that is,

$$\frac{\delta (f \circ \varphi)}{\delta \alpha} = \mathbf{D}\varphi(\alpha)^* \cdot \frac{\delta f}{\delta \gamma}. \tag{2.4.16}$$

Similarly if $\psi : \mathbb{R} \to \mathbb{R}$ is differentiable then for $\alpha, \beta \in \mathbf{F}$,

$$\mathbf{D}(\psi \circ f)(\alpha) \cdot \beta = \mathbf{D}\psi(f(\alpha)) \cdot (\mathbf{D} f(\alpha) \cdot \beta)$$

where the first dot on the right hand side is ordinary multiplication by $\mathbf{D}\psi(f(a)) \in \mathbb{R}$. Hence

$$\left\langle \frac{\delta (\psi \circ f)}{\delta \alpha}, \beta \right\rangle = \mathbf{D}\psi(f(\alpha)) \circ \left\langle \frac{\delta f}{\delta \alpha}, \beta \right\rangle = \left\langle \psi'(f(\alpha)) \frac{\delta f}{\delta \alpha}, \beta \right\rangle$$

that is,

$$\frac{\delta (\psi \circ f)}{\delta \alpha} = \psi'(f(\alpha)) \frac{\delta f}{\delta \alpha}. \tag{2.4.17}$$

Extrema for Real Valued Functions on Banach Spaces. Much of this theory proceeds in a manner parallel to calculus.

2.4.26 Definition. *Let $f : U \subset \mathbf{E} \to \mathbb{R}$ be a continuous function, U open in $\mathbf{E}$. We say f has a **local minimum** (resp., **maximum** at $u_0 \in U$, if there is a neighborhood V of $u_0, V \subset U$ such that $f(u_0) \leq f(u)$ (resp., $f(u_0) \geq f(u)$) for all $u \in V$. If the inequality is strict, u_0 is called a **strict local minimum** (resp., **maximum**). The point u_0 is called a **global minimum** (resp., **maximum**) if $f(u_0) \leq f(u)$ (resp., $f(u_0) \geq f(u)$) for all $u \in U$. Local maxima and minima are called **local extrema**.*

2.4.27 Proposition. *Let $f : U \subset \mathbf{E} \to \mathbb{R}$ be a continuous function differentiable at $u_0 \in U$. If f has a local extremum at u_0, then $\mathbf{D}f(u_0) = 0$.*

Proof. If u_0 is a local minimum, then there is a neighborhood V of U such that $f(u_0 + th) - f(u_0) \geq 0$ for all $h \in V$. Therefore, the limit of $[f(u_0 + th) - f(u_0)]/t$ as $t \to 0$, $t \geq 0$ is ≥ 0 and as $t \to 0$, $t \leq 0$ is ≤ 0. Since both limits equal $\mathbf{D}f(u_0)$, it must vanish. ■

This criterion is not sufficient as the elementary calculus example $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ shows. Also, if U is not open, the values of f on the boundary of U must be examined separately.

2.4.28 Proposition. *Let $f : U \subset \mathbf{E} \to \mathbb{R}$ be twice differentiable at $u_0 \in U$.*

(i) *If u_0 is a local minimum (maximum), then $\mathbf{D}^2 f(u_0) \cdot (e, e) \geq 0$ (≤ 0) for all $e \in \mathbf{E}$.*

(ii) *If u_0 is a non-degenerate critical point f, that is, $\mathbf{D}f(u_0) = 0$ and $\mathbf{D}^2 f(u_0)$ defines an isomorphism of $\mathbf{E}$ with $\mathbf{E}^*$, and if $\mathbf{D}^2 f(u_0) \cdot (e, e) > 0$ (< 0) for all $e \neq 0$, $e \in \mathbf{E}$, then u_0 is a strict local minimum (maximum) of f.*

Proof. (i) By Taylor's formula, in a neighborhood V of U_0,

$$0 \leq f(u_0 + h) - f(u_0) = \frac{1}{2}\mathbf{D}f(u_0)(h, h) + o(h^2)$$

for all $h \in V$. If $e \in \mathbf{E}$ is arbitrary, for small $t \in \mathbb{R}$, $te \in V$, so that

$$0 \leq \frac{1}{2}\mathbf{D}^2 f(u_0)(te, te) + o(t^2 e^2)$$

implies

$$\mathbf{D}^2 f(u_0)(e, e) + \frac{2}{t^2} o(t^2 e^2) \geq 0.$$

Now let $t \to 0$.

(ii) Denote by $T : \mathbf{E} \to \mathbf{E}^*$ the isomorphism defined by $e \mapsto \mathbf{D}^2 f(u_0) \cdot (e, \cdot)$, so that there exists $a > 0$ such that

$$a\|e\| \leq \|Te\| = \sup_{\|e'\|=1} |\langle Te, e' \rangle| = \sup_{\|e'\|=1} |\mathbf{D}^2 f(u_0) \cdot (e, e')|.$$

By hypothesis and symmetry of the second derivative,

$$\begin{aligned} 0 &< \mathbf{D}^2 f(u_0) \cdot (e + se', e + se') \\ &= s^2 \mathbf{D}^2 f(u_0) \cdot (e', e') + 2s\mathbf{D}^2 f(u_0) \cdot (e, e') + \mathbf{D}^2 f(u_0) \cdot (e, e) \end{aligned}$$

which is a quadratic form in s. Therefore its discriminant must be negative, that is,

$$\begin{aligned} |\mathbf{D}^2 f(u_0) \cdot (e, e')|^2 &< \mathbf{D}^2 f(u_0) \cdot (e', e')\mathbf{D}^2 f(u_0) \cdot (e, e) \\ &\leq \|\mathbf{D}^2 f(u_0)\| \mathbf{D}^2 f(u_0) \cdot (e, e), \end{aligned}$$

and we get

$$a\|e\| \leq \sup_{\|e'\|=1} |\mathbf{D}^2 f(u_0) \cdot (e, e')| \leq \|\mathbf{D}^2 f(u_0)\|^{1/2} [\mathbf{D}^2 f(u_0) \cdot (e, e)]^{1/2}.$$

Therefore, letting $m = a^2 / \|\mathbf{D}^2 f(u_0)\|$, the following inequality holds for any $e \in \mathbf{E}$:

$$\mathbf{D}^2 f(u_0) \cdot (e, e) \geq m\|e\|^2.$$

Thus, by Taylor's theorem we have

$$f(u_0+h)-f(u_0)=\frac{1}{2}\mathbf{D}^2f(u_0)\cdot(h,h)+o(h^2)\geq\frac{m\|h\|^2}{2}+o(h^2).$$

Let $\varepsilon>0$ be such that if $\|h\|<\varepsilon$, then $|o(h^2)|\leq m\|h\|^2/4$, which implies $f(u_0+h)-f(u_0)\geq m\|h\|^2/4>0$ for $h\neq 0$, and thus u_0 is a strict local minimum of f.

■

The condition in (i) is not sufficient for f to have a local minimum at u_0. For example, $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=x^2-y^4$ has $f(0,0)=0$, $\mathbf{D}f(0,0)=0$, $\mathbf{D}^2f(0,0)\cdot(x,y)^2=2x^2\geq 0$ and in any neighborhood of the origin, f changes sign. The conditions in (ii) are not necessary for f to have a strict local minimum at u_0. For example $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^4$ has $f(0)=f'(0)=f''(0)=f'''(0)=0$, $f^{(4)}(0)>0$ and 0 is a strict global minimum for f. On the other hand, if the conditions in (ii) hold and u_0 is the only critical point of a differentiable function $f:U\to\mathbb{R}$, then u_0 is a strict global minimum of f. For if there was another point $u_1\in U$ with $f(u_1)\leq f(u_0)$ on the straight line segment $(1-t)u_0+tu_1$, $t\in[0,1]$ there exists a point u_2 such that $f(u_2)>f(u_0)\geq f(u_1)$ since by (ii) u_0 is a strict local minimum. Therefore, there exists u_3 on this segment, $u_3\neq u_0$, u_1 such that $f(u_3)=f(u_0)$. But then by the mean value theorem (Proposition 2.4.8) there exists $u_4\neq u_0,u_3$ such that $\mathbf{D}f(u_4)=0$ which contradicts uniqueness of the critical point. Finally, care has to be taken with the statement in (ii): non-degeneracy holds in the topology of $\mathbf{E}$. If $\mathbf{E}$ is continuously embedded in another Banach space $\mathbf{F}$ and $\mathbf{D}^2f(u_0)$ is non-degenerate in $\mathbf{F}$ only, u_0 need not even be a minimum. For example, consider the smooth map

$$f:L^4([0,1])\to\mathbb{R},\quad f(u)=\frac{1}{2}\int_0^1(u(x)^2-u(x)^4)\,dx$$

and note $f(0)=0$, $\mathbf{D}f(0)=0$, and

$$\mathbf{D}^2f(0)(v,v)=\int_0^1 v(x)^2\,dx>0\quad\text{for } v\neq 0,$$

and that $\mathbf{D}^2f(0)$ defines an isomorphism of $L^4([0,1])$ with $L^{exc:4.3-27}([0,1])$. Alternatively, $\mathbf{D}^2f(0)$ is non-degenerate on $L^2([0,1])$ *not* on $L^4([0,1])$. Also note that in any neighborhood of 0 in $L^4([0,1])$, f changes sign: $f(1/n)=(n^2-1)/2n^4\geq 0$ for $n\geq 2$, but $f(u_n)=-12/n<0$ for $n\geq 1$ if

$$u_n=\begin{cases}2, & \text{on } [0,1/n];\\ 0, & \text{elsewhere}\end{cases}$$

and both $1/n,u_n$ converge to 0 in $L^4([0,1])$. Thus, even though $\mathbf{D}^2f(0)$ is positive, 0 is not a minimum of f. (See Ball and Marsden [1984] for more sophisticated examples of this sort.)

Exercises

◇ **2.4-1.** Show that if $g:U\subset\mathbf{E}\to L(\mathbf{F},\mathbf{G})$ is C^r, then $f:U\times\mathbf{F}\to\mathbf{G}$, defined by $f(u,v)=(g(u))(v)$, $u\in U$, $v\in\mathbf{F}$ is also C^r.
Hint: Apply the Leibniz rule with $L(\mathbf{F},\mathbf{G})\times\mathbf{F}\to\mathbf{G}$ the evaluation map.

◇ **2.4-2.** Show that if $f:U\subset\mathbf{E}\to L(\mathbf{F},\mathbf{G})$, $g:U\subset\mathbf{E}\to L(\mathbf{G},\mathbf{H})$ are C^r mappings then so is $h:U\subset\mathbf{E}\to L(\mathbf{F},\mathbf{H})$, defined by $h(u)=g(u)\circ f(u)$.

◇ **2.4-3.** Extend Leibniz' rule to multilinear mappings and find a formula for the derivative.

◇ **2.4-4.** Define a map $f:U\subset\mathbf{E}\to\mathbf{F}$ to be of ***class*** T^1 if it is differentiable, its tangent map $T^f:U\times\mathbf{E}\to\mathbf{F}\times\mathbf{F}$ is continuous and $\|\mathbf{D}f(x)\|$ is locally bounded.

(i) For $\mathbf{E}$ and $\mathbf{F}$ finite dimensional, show that this is equivalent to C^1.

(ii) (Project.) Investigate the validity of the chain rule and Taylor's theorem for T^r maps .

(iii) (Project.) Show that the function developed in Smale [1964] is T^2 but is *not* C^2.

◇ **2.4-5.** Suppose that $f : \mathbf{E} \to \mathbf{F}$ (where $\mathbf{E}, \mathbf{F}$ are real Banach spaces) is ***homogeneous of degree*** k (where k is a nonnegative integer). That is, $f(te) = t^k f(e)$ for all $t \in \mathbb{R}$, and $e \in \mathbf{E}$.

(i) Show that if f is differentiable, then $\mathbf{D}f(u) \cdot u = kf(u)$.

HINT: Let $g(t) = f(tu)$ and compute dg/dt.

(ii) If $\mathbf{E} = \mathbb{R}^n$ and $\mathbf{F} = \mathbb{R}$, show that this relation is equivalent to

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = kf(x)$$

Show that maps multilinear in k variables are homogeneous of degree k. Give other examples.

(iii) If f is C^k show that $f(e) = (1/k!)\mathbf{D}^k f(0) \cdot e^k$, that is, f may be regarded as an element of $S^k(\mathbf{E}, \mathbf{F})$ and thus it is C^∞.

HINT: $f(0) = 0$; inductively applying Taylor's theorem and replacing at each step h by th, show that

$$f(h) = \frac{1}{k!}\mathbf{D}^k f(0) \cdot h^k + \frac{1}{t^k} o(t^k h^k).$$

◇ **2.4-6.** Let $e_1, \ldots, e_{n-1} \in \mathbf{E}$ be fixed and $f : U \subset \mathbf{E} \to \mathbf{F}$ be n times differentiable. Show that the map $g : U \subset \mathbf{E} \to \mathbf{F}$ defined by $g(u) = \mathbf{D}^{n-1} f(u) \cdot (e_1, \ldots, e_{n-1})$ is differentiable and

$$\mathbf{D}g(u) \cdot e = \mathbf{D}^n f(u) \cdot (e, e_1, \ldots, e_{n-1}).$$

◇ **2.4-7.** (i) Prove the following refinement of Proposition 2.4.14. If f is C^1 and $\mathbf{D}_1\mathbf{D}_2 f(u)$ exists and is continuous in u, then $\mathbf{D}_2\mathbf{D}_1 f(u)$ exists and these are equal.

(ii) The hypothesis in (i) cannot be weakened: show that the function

$$f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

is C^1, has $\partial^2 f/\partial x \partial y$, $\partial^2 f/\partial y \partial x$ continuous on $\mathbb{R}^2/\{(0, 0)\}$, but that $\partial^2 f(0, 0)/\partial x \partial y \neq \partial^2 f(0, 0)/\partial y \partial x$.

◇ **2.4-8.** For $f : U \subset \mathbf{E} \to \mathbf{F}$, show that the second tangent map is given as follows:

$$\begin{aligned} T^2 f : (U \times \mathbf{E}) \times (\mathbf{E} \times \mathbf{E}) \to (\mathbf{F} \times \mathbf{F}) \times (\mathbf{F} \times \mathbf{F})&(u, e_1, e_2, e_3) \\ \mapsto (f(u), \mathbf{D}f(u) \cdot e_1, &\mathbf{D}f(u) \cdot e_2, \\ &\mathbf{D}^2 f(u) \cdot (e_1, e_2) + \mathbf{D}f(u) \cdot e_3). \end{aligned}$$

◇ **2.4-9.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = 2x^2 y/(x^4 + y^2)$ if $(x, y) \neq (0, 0)$ and 0 if $(x, y) = (0, 0)$. Show that

(i) f is discontinuous at $(0, 0)$, hence is not differentiable at $(0, 0)$;

(ii) all directional derivatives exist at $(0, 0)$; that is, f is Gâteaux differentiable.

◇ **2.4-10** (Differentiating sequences). Let $f_n : U \subset \mathbf{E} \to \mathbf{F}$ be a sequence of C^r maps, where $\mathbf{E}$ and $\mathbf{F}$ are Banach spaces. If $\{f_n\}$ converges pointwise to $f : U \to \mathbf{F}$ and if $\{\mathbf{D}^j f_n\}$, $0 \leq j \leq r$, converges locally uniformly to a map $g^j : U \to L^j_s(\mathbf{E}, \mathbf{F})$, then show that f is C^r, $\mathbf{D}^j f = g^j$ and $\{f_n\}$ converges locally uniformly to f.
HINT: For $r = 1$ use the mean value inequality and continuity of g^1 to conclude that

$$\begin{aligned}\|f(u+h) - f(u) - g^1(u) \cdot h\| &\leq \|f(u+h) - f_n(u+h) - [f(u) - f_n(u)]\| \\ &\quad + \|f_n(u+h) - f_n(u) - \mathbf{D}f_n(u) \cdot h\| \\ &\quad + \|\mathbf{D}f_n(u) \cdot h - g^1(u) \cdot h\| \\ &\leq e\|h\|.\end{aligned}$$

For general r use the converse to Taylor's theorem.

◇ **2.4-11** (α Lemma). In the context of Lemma 2.4.18 let $\alpha(g) = g \circ f$. Show that α is continuous linear and hence is C^∞.

◇ **2.4-12.** Consider the map $\Phi : C^1([0,1]) \to C^0([0,1])$ given by $\Phi(f)(x) = \exp[f'(x)]$. Show that Φ is C^∞ and compute $\mathbf{D}\Phi$.

◇ **2.4-13** (Whitney [1943a]). Let $f : U \subset \mathbf{E} \to \mathbf{F}$ be of class C^{k+p} with Taylor expansion

$$\begin{aligned}f(b) &= f(a) + \mathbf{D}f(a) \cdot (b-a) + \cdots + \frac{1}{k!}\mathbf{D}^k f(a) \cdot (b-a)^k \\ &\quad + \left\{\int_0^1 \frac{(1-t)^{k-1}}{(k-1)!}[\mathbf{D}^k f((1-t)a + tb) - \mathbf{D}^k f(a)]\, dt\right\} \cdot (b-a)^k.\end{aligned}$$

(i) Show that the remainder $R_k(a,b)$ is C^{k+p} for $b \neq a$ and C^p for $a, b \in \mathbf{E}$. If $\mathbf{E} = \mathbf{F} = \mathbb{R}$, $R_k(a,a) = 0$, and

$$\lim_{b \to a}(|b-a|^i \mathbf{D}^{i+p} R_k(a,b)) = 0, \quad 1 \leq i \leq k.$$

(For generalizations to Banach spaces, see Tuan and Ang [1979].)

(ii) Show that the conclusion in (i) cannot be improved by considering $f(x) = |x|^{k+p+1/2}$.

◇ **2.4-14** (Whitney [1943b]). Let $f : \mathbb{R} \to \mathbb{R}$ be an even (resp., odd) function; that is, $f(x) = f(-x)$ (resp., $f(x) = -f(-x)$).

(i) Show that $f(x) = g(x^2)$ (resp., $f(x) = xg(x^2)$) for some g.

(ii) Show that if f is C^{2k} (resp., C^{2k+1}) then g is C^k
HINT: Use the converse to Taylor's theorem.

(iii) Show that (ii) is still true if $k = \infty$.

(iv) Let $f(x) = |x|^{2k+1+1/2}$ to show that the conclusion in (ii) cannot be sharpened.

◇ **2.4-15** (Buchner, Marsden, and Schecter [1983b]). Let $\mathbf{E} = L^4([0,1])$ and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $\varphi'(\lambda) = 1$, if $-1 \leq \lambda \leq 1$ and $\varphi'(\lambda) = 0$, if $|\lambda| \geq 2$. Assume φ is monotone increasing with $\varphi = -M$ for $\lambda \leq -2$ and $\varphi = M$ for $\lambda \geq 2$. Define the map $h : \mathbf{E} \to \mathbb{R}$ by

$$h(u) = \frac{1}{3}\int_0^1 \varphi([u(x)]^3)\, dx.$$

(i) Show that h is C^3 using the converse to Taylor's theorem.

HINT: Let $\psi(\lambda) = \varphi(\lambda^3)$, write out Taylor's theorem for $r = 3$ for $\psi(\lambda)$, and plug in $u(x)$ for λ.

(ii) The formal L^2 gradient of h (i.e., the functional derivative $\delta h/\delta u$) is given by

$$\nabla h(u) = \frac{1}{3}\psi'(u),$$

where $\psi(\lambda) = \varphi(\lambda^3)$. Show that $\nabla h : \mathbf{E} \to \mathbf{E}$ is C^0 but is not C^1.

HINT: Its derivative would be $v \mapsto \psi''(u)v/3$. Let $a \in [0,1]$ be such that $\varphi''(a)/3 \neq 0$ and let $u_n = a$ on $[0,1/n]$, $u_n = 0$ elsewhere; $v_n = n^{exc:1.4-27}$ on $[0,1/n]$, $v_n = 0$ elsewhere. Show that in $L^4([0,1])$, $u_n \to 0$, $\|v_n\| = 1$, $\psi''(u_n) \cdot v_n$ does not converge to 0, but $\psi''(0) = 0$. Using the same method, show h is not C^4 on $L^4([0,1])$.

(iii) Show: if q is a positive integer and $\mathbf{E} = L^q([0,1])$, then h is C^{q-1} but is not C^q.

(iv) Let

$$f(u) = \frac{1}{2}\int_0^1 |u(x)|^2\,dx + h(u)$$

Show that on $L^4([0,1])$, f has a formally non-degenerate critical point at 0 (i.e., $\mathbf{D}^2 f(0)$ defines an isomorphism of $L^2([0,1])$), yet this critical point is *not isolated*.

HINT: Consider the function $u_n = -1$ on $[0,1/n]$; 0 on $]1/n,1]$. This exercise is continued in Exercise 5.4-8.

◇ **2.4-16.** Let $\mathbf{E}$ be the space of maps $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ with $\mathbf{A}(x) \to 0$ as $x \to 0$ sufficiently rapidly. Let $f : \mathbf{E} \to \mathbb{R}$ and show

$$\frac{\delta}{\delta \mathbf{A}} f(\operatorname{curl}\mathbf{A}) = \operatorname{curl}\frac{\delta f}{\delta \mathbf{A}}.$$

HINT: Specify whatever smoothness and fall-off hypotheses you need; use $\mathbf{A}\cdot\operatorname{curl}\mathbf{B} - \mathbf{B}\cdot\operatorname{curl}\mathbf{A} = \operatorname{div}(\mathbf{B}\times\mathbf{A})$, the divergence theorem, and the chain rule.

◇ **2.4-17.** (i) Let $\mathbf{E} = \{\, \mathbf{B} \mid \mathbf{B}$ is a vector field on $\mathbb{R}^3$ vanishing at ∞ and such that $\operatorname{div} B = 0 \,\}$ and pair $\mathbf{E}$ with itself via $\langle \mathbf{B}, \mathbf{B}' \rangle = \int \mathbf{B}(x)\cdot\mathbf{B}'(x)\,dx$. Compute $\delta F/\delta \mathbf{B}$, where F is defined by $F = (1/2)\int \|\mathbf{B}\|^2 d^3x$.

(ii) Let $\mathbf{E} = \{\, \mathbf{B} \mid \mathbf{B}$ is a vector field on $\mathbb{R}^3$ vanishing at ∞ such that $\mathbf{B} = \nabla \times \mathbf{A}$ for some $\mathbf{A} \,\}$ and let

$$\mathbf{F} = \{\, \mathbf{A}' \mid \mathbf{A}' \text{ is a vector field on } \mathbb{R}^3,\ \operatorname{div}\mathbf{A}' = 0 \,\}$$

with the pairing $\langle \mathbf{B}, \mathbf{A}' \rangle = \int \mathbf{A}\cdot\mathbf{A}' d^3x$. Show that this pairing is well defined. Compute $\delta F/\delta \mathbf{B}$, where F is as in (i) . Why is your answer different?

2.5 The Inverse and Implicit Function Theorems

The inverse and implicit function theorems are pillars of nonlinear analysis and geometry, so we give them special attention in this section. Throughout, $\mathbf{E}, \mathbf{F}, \ldots,$ are assumed to be Banach spaces. In the finite-dimensional case these theorems have a long and complex history; the infinite-dimensional version is apparently due to Hildebrandt and Graves [1927].

The Inverse Function Theorem. This theorem states that if the linearization of the equation $f(x) = y$ is uniquely invertible, then locally so is f; that is, we can uniquely solve $f(x) = y$ for x as a function of y. To formulate the theorem, the following terminology is useful.

2.5.1 Definition. *A map $f : U \subset \mathbf{E} \to V \subset \mathbf{F}$, where U and V are open subsets of $\mathbf{E}$ and $\mathbf{F}$ respectively, is a C^r* ***diffeomorphism*** *if f is of class C^r, is a bijection (i.e., f is one-to-one and onto from U to V), and f^{-1} is also of class C^r.*

The example $f(x) = x^3$ shows that a map can be smooth and bijective, but its inverse need not be smooth. A theorem guaranteeing a smooth inverse is the following.

2.5.2 Theorem (Inverse Mapping Theorem). *Let $f : U \subset \mathbf{E} \to \mathbf{F}$ be of class C^r, $r \geq 1$, $x_0 \in U$, and suppose that $\mathbf{D}f(x_0)$ is a linear isomorphism. Then f is a C^r diffeomorphism of some neighborhood of x_0 onto some neighborhood of $f(x_0)$ and, moreover, the derivative of the inverse function is given by*

$$\mathbf{D}f^{-1}(y) = [\mathbf{D}f(f^{-1}(y))]^{-1}$$

for y in this neighborhood of $f(x_0)$.

Although our immediate interest is the finite-dimensional case, for Banach spaces it is good to keep in mind the *Banach isomorphism theorem*: If $T : \mathbf{E} \to \mathbf{F}$ is linear, bijective, and continuous, then T^{-1} is continuous. (See Theorem 2.2.19.)

Proof of the Inverse Function Theorem. To prove the theorem, we assemble a few lemmas. First recall the contraction mapping principle from §1.2.

2.5.3 Lemma. *Let M be a complete metric space with distance function $d : M \times M \to \mathbb{R}$. Let $F : M \to M$ and assume there is a constant λ, $0 \leq \lambda < 1$, such that for all $x, y \in M$,*

$$d(F(x), F(y)) \leq \lambda d(x, y).$$

Then F has a unique fixed point $x_0 \in M$; that is, $F(x_0) = x_0$.

This result is the basis of many important existence theorems in analysis. The other fundamental fixed point theorem in analysis is the Schauder fixed point theorem, which states that a continuous map of a compact convex set (in a Banach space, say) to itself, has a fixed point—not necessarily unique, however.

2.5.4 Lemma. *The set $\mathrm{GL}(\mathbf{E}, \mathbf{F})$ of linear isomorphisms from $\mathbf{E}$ to $\mathbf{F}$ is open in $L(\mathbf{E}, \mathbf{F})$.*

Proof. We can assume $\mathbf{E} = \mathbf{F}$. Indeed, if $\varphi_0 \in \mathrm{GL}(\mathbf{E}, \mathbf{F})$, the linear map $\psi \mapsto \varphi_0^{-1} \circ \psi$ from $L(\mathbf{E}, \mathbf{F})$ to $L(\mathbf{E}, \mathbf{E})$ is continuous and $\mathrm{GL}(\mathbf{E}, \mathbf{F})$ is the inverse image of $\mathrm{GL}(\mathbf{E}, \mathbf{E})$. Let

$$\|\alpha\| = \sup_{\substack{e \in \mathbf{E} \\ \|e\|=1}} \|\alpha(e)\|$$

be the operator norm on $L(\mathbf{E}, \mathbf{F})$ relative to given norms on $\mathbf{E}$ and $\mathbf{F}$. For $\varphi \in \mathrm{GL}(\mathbf{E}, \mathbf{E})$, we need to prove that ψ sufficiently near φ is also invertible. We will show that

$$\|\psi - \varphi\| < \|\varphi^{-1}\|^{-1}$$

implies $\psi \in \mathrm{GL}(\mathbf{E}, \mathbf{E})$. The key is that $\| \cdot \|$ is an algebra norm. That is,

$$\|\beta \circ \alpha\| \leq \|\beta\| \, \|\alpha\|$$

for $\alpha \in L(\mathbf{E}, \mathbf{E})$ and $\beta \in L(\mathbf{E}, \mathbf{E})$ (see §2.2). Since

$$\psi = \varphi \circ (I - \varphi^{-1} \circ (\varphi - \psi)),$$

φ is invertible, and our norm assumption shows that

$$\|\varphi^{-1} \circ (\varphi - \psi)\| < 1,$$

it is sufficient to show that $I - \xi$ is invertible whenever $\|\xi\| < 1$. (I is the identity operator.) Consider the following sequence called the ***Neumann series***:

$$\begin{aligned}
\xi_0 &= I, \\
\xi_1 &= I + \xi, \\
\xi_2 &= I + \xi + \xi \circ \xi, \\
&\ \ \vdots \\
\xi_n &= I + \xi + \xi \circ \xi + \cdots + (\xi \circ \xi \circ \cdots \circ \xi).
\end{aligned}$$

Using the triangle inequality and $\|\xi\| < 1$, we can compare this sequence to the sequence of real numbers, $1, 1 + \|\xi\|, 1 + \|\xi\| + \|\xi\|^2, \ldots$, which is a Cauchy sequence since the geometric series $\sum_{n=0}^{\infty} \|\xi\|^n$ converges. Because $L(\mathbf{E}, \mathbf{E})$ is complete, ξ_n is a convergent sequence. The limit, say ρ, is the inverse of $I - \xi$ because $(I - \xi)\xi_n = I - (\xi \circ \xi \circ \cdots \circ \xi)$, so letting $n \to \infty$, we get $(I - \xi)\rho = I$. ■

2.5.5 Lemma. *Let $\mathfrak{I} : \mathrm{GL}(\mathbf{E}, \mathbf{F}) \to \mathrm{GL}(\mathbf{F}, \mathbf{E})$ be given by $\varphi \mapsto \varphi^{-1}$. Then $\mathfrak{I}$ is of class C^∞ and*

$$\mathbf{D}\mathfrak{I}(\varphi) \cdot \psi = -\varphi^{-1} \circ \psi \circ \varphi^{-1}.$$

(*For $\mathbf{D}^r\mathfrak{I}$, see Supplement* 2.5E.)

Proof. We may assume $\mathrm{GL}(\mathbf{E}, \mathbf{F}) \neq \varnothing$. We claim that $\mathfrak{I}$ is differentiable and that $\mathbf{D}\mathfrak{I}(\varphi) \cdot \psi = -\varphi^{-1} \circ \psi \circ \varphi^{-1}$, then it will follow from Leibniz' rule that $\mathfrak{I}$ is of class C^∞. Indeed $\mathbf{D}\mathfrak{I} = B(\mathfrak{I}, \mathfrak{I})$ where $B \in L^2(L(\mathbf{F}, \mathbf{E}); L(L(\mathbf{E}, \mathbf{F}), L(\mathbf{F}, \mathbf{E})))$ is defined by $B(\psi_1, \psi_2)(A) = -\psi_1 \circ A \circ \psi_2$, where $\psi_1, \psi_2 \in L(\mathbf{F}, \mathbf{E})$ and $A \in L(\mathbf{E}, \mathbf{F})$, which shows inductively that if $\mathfrak{I}$ is C^k then it is C^{k+1}.

To show our claim that $\mathfrak{I}$ is differentiable, we use the definition of differentiability. Since the map $\psi \mapsto -\varphi^{-1} \circ \psi \circ \varphi^{-1}$ is linear ($\psi \in L(\mathbf{E}, \mathbf{F})$), we must show that

$$\lim_{\psi \to \varphi} \frac{\|\psi^{-1} - (\varphi^{-1} - \varphi^{-1} \circ \psi \circ \varphi^{-1} + \varphi^{-1} \circ \varphi \circ \varphi^{-1})\|}{\|\psi - \varphi\|} = 0.$$

Note that

$$\begin{aligned}
\psi^{-1} - (\varphi^{-1} &- \varphi^{-1} \circ \psi \circ \varphi^{-1} + \varphi^{-1} \circ \varphi \circ \varphi^{-1}) \\
&= \psi^{-1} - 2\varphi^{-1} + \varphi^{-1} \circ \psi \circ \varphi^{-1} \\
&= \psi^{-1} \circ (\psi - \varphi) \circ \varphi^{-1} \circ (\psi - \varphi) \circ \varphi^{-1}.
\end{aligned}$$

Again, using $\|\beta \circ \alpha\| \leq \|\alpha\| \, \|\beta\|$ for $\alpha \in L(\mathbf{E}, \mathbf{F})$ and $\beta \in L(\mathbf{F}, \mathbf{G})$, we get

$$\|\psi^{-1} \circ (\psi - \varphi) \circ \varphi^{-1} \circ (\psi - \varphi) \circ \varphi^{-1}\| \leq \|\psi^{-1}\| \, \|\psi - \varphi\|^2 \|\varphi^{-1}\|^2.$$

With this inequality, the limit is clearly zero. ■

Proof of the Inverse Mapping Theorem. We claim that it is enough to prove it under the simplifying assumptions $x_0 = 0$, $f(x_0) = 0$, $\mathbf{E} = \mathbf{F}$, and $\mathbf{D}f(0)$ is the identity. Indeed, replace f by

$$h(x) = \mathbf{D}f(x_0)^{-1} \circ [f(x + x_0) - f(x_0)].$$

Let $g(x) = x - f(x)$ so $\mathbf{D}g(0) = 0$. Choose $r > 0$ so that $\|x\| \leq r$ implies $\|\mathbf{D}g(x)\| \leq 1/2$, which is possible by continuity of $\mathbf{D}g$. Thus, by the mean value inequality, $\|x\| \leq r$ implies $\|g(x)\| \leq r/2$. Let

$$B_\varepsilon(0) = \{\, x \in \mathbf{E} \mid \|x\| \leq e \,\}.$$

For $y \in B_{r/2}(0)$, let $g_y(x) = y + g(x)$. If $y \in B_{r/2}(0)$ and $x_1, x_2 \in B_r(0)$, then $\|y\| \le r/2$ and $\|g(x)\| \le r/2$, so

$$\|g_y(x)\| \le \|y\| + \|g(x)\| \le r, \tag{i}$$

and, by the mean value inequality,

$$\|g_y(x_1) - g_y(x_2)\| \le \frac{\|x_1 - x_2\|}{2}. \tag{ii}$$

This shows that for y in the ball of radius $r/2$, g_y maps the closed ball (a complete metric space) of radius r to itself and is a contraction. Thus by the contraction mapping theorem (Lemma 2.5.3), g_y has a unique fixed point x in $B_r(0)$. This point x is the unique solution of $f(x) = y$. Thus f has an inverse

$$f^{-1} : V_0 = D_{r/2}(0) \to U_0 = f^{-1}(D_{r/2}(0)) \subset D_r(0).$$

From (ii) with $y = 0$, we have $\|(x_1 - f(x_1)) - (x_2 - f(x_2))\| \le \|x_1 - x_2\|/2$, and so

$$\|x_1 - x_2\| - \|f(x_1) - f(x_2)\| \le \frac{\|x_1 - x_2\|}{2},$$

that is,

$$\|x_1 - x_2\| \le 2\|f(x_1) - f(x_2)\|.$$

Thus we have

$$\|f^{-1}(y_1) - f^{-1}(y_2)\| \le 2\|y_1 - y_2\|, \tag{iii}$$

so f^{-1} is continuous.

From Lemma 2.5.4 we can choose r small enough so that $\mathbf{D}f(x)^{-1}$ exists for $x \in D_r(0)$. Moreover, by continuity, $\|\mathbf{D}f(x)^{-1}\| \le M$ for some M and all $x \in D_r(0)$ can be assumed as well. If $y_1, y_2 \in D_{r/2}(0)$, $x_1 = f^{-1}(y_1)$, and $x_2 = f^{-1}(y_2)$, then

$$\begin{aligned}
&\|f^{-1}(y_1) - f^{-1}(y_2) - \mathbf{D}f(x_2)^{-1} \cdot (y_1 - y_2)\| \\
&\quad = \|x_1 - x_2 - \mathbf{D}f(x_2)^{-1} \cdot [f(x_1) - f(x_2)]\| \\
&\quad = \|\mathbf{D}f(x_2)^{-1} \cdot \{\mathbf{D}f(x_2) \cdot (x_1 - x_2) - f(x_1) + f(x_2)\}\| \\
&\quad \le M\|f(x_1) - f(x_2) - \mathbf{D}f(x_2) \cdot (x_1 - x_2)\|.
\end{aligned}$$

This, together with (iii), shows that f^{-1} is differentiable with derivative $\mathbf{D}f(x)^{-1}$ at $f(x)$; that is, $\mathbf{D}(f^{-1}) = I \circ \mathbf{D}f \circ f^{-1}$ on $V_0 = D_{r/2}(0)$. This formula, the chain rule, and Lemma 2.5.5 show inductively that if f^{-1} is C^{k-1} then f^{-1} is C^k for $1 \le k \le r$. ■

This argument also proves the following: *if $f : U \to V$ is a C^r homeomorphism where $U \subset \mathbf{E}$ and $V \subset \mathbf{F}$ are open sets, and $\mathbf{D}f(u) \in \mathrm{GL}(\mathbf{E}, \mathbf{F})$ for $u \in U$, then f is a C^r diffeomorphism.*

For a Lipschitz inverse function theorem see Exercise 2.5-11.

Supplement 2.5A

The Size of the Neighborhoods in the Inverse Mapping Theorem

An analysis of the preceding proof also gives *explicit estimates* on the size of the ball on which $f(x) = y$ is solvable. Such estimates are sometimes useful in applications. The easiest one to use in examples involves estimates on the second derivative.[2]

[2]We thank M. Buchner for his suggestions concerning this supplement.

2.5.6 Proposition. *Suppose $f : U \subset \mathbf{E} \to \mathbf{F}$ is of class C^r, $r \geq 2$, $x_0 \in U$, and $\mathbf{D}f(x_0)$ is an isomorphism. Let*

$$L = \|\mathbf{D}f(x_0)\| \quad \textit{and} \quad M = \|\mathbf{D}f(x_0)^{-1}\|.$$

Assume

$$\|\mathbf{D}^2 f(x)\| \leq K \quad \textit{for } \|x - x_0\| \leq R \textit{ and } B_R(x_0) \subset U.$$

Let

$$P = \min\left(\frac{1}{2KM}, R\right), \quad Q = \min\left(\frac{1}{2NL}, \frac{P}{M}, P\right),$$

and

$$S = \min\left(\frac{1}{2KM}, \frac{Q}{2L}, Q\right).$$

Here $N = 8M^3K$. Then f maps an open set $G \subset D_P(x_0)$ diffeomorphically onto $D_{P/2M}(y_0)$ and f^{-1} maps an open set $H \subset D_Q(y_0)$ diffeomorphically onto $D_{Q/2L}(x_0)$. Moreover, $B_{Q/2L}(x_0) \subset G \subset D_P(x_0)$ and $B_{S/2M}(y_0) \subset H \subset D_Q(y_0) \subset D_{P/2M}(y_0)$. See Figure 2.5.1.

Proof. We can assume $x_0 = 0$ and $f(x_0) = 0$. From

$$\begin{aligned} \mathbf{D}f(x) = \mathbf{D}f(0) &= \int_0^1 \mathbf{D}(\mathbf{D}f(tx)) \cdot x \, dt \\ &= \mathbf{D}f(0) \cdot \left\{ I + [\mathbf{D}f(0)]^{-1} \cdot \int_0^1 \mathbf{D}^2 f(tx) \cdot x \, dt \right\} \end{aligned}$$

and the fact that

$$\|(I + A)^{-1}\| \leq 1 + \|A\| + \|A\|^2 + \cdots = \frac{1}{1 - \|A\|}$$

for $\|A\| < 1$ (see the proof of Lemma 2.5.5), we get

$$\|\mathbf{D}f(x)^{-1}\| \leq 2M \quad \text{if } \|x\| \leq R \text{ and } \|x\| \leq \frac{1}{2MK},$$

that is, if $\|x\| \leq P$.
As in the proof of the inverse function theorem, let $g_y(x) = [\mathbf{D}f(0)]^{-1} \cdot (y + \mathbf{D}f(0)x - f(x))$. Write

$$\begin{aligned} \varphi(x) &= \mathbf{D}f(0) \cdot x - f(x) \\ &= \int_0^1 \mathbf{D}\varphi(sx) \cdot x \, ds \\ &= -\int_0^1 \int_0^1 \mathbf{D}^2 f(tsx) \cdot (sx, x) \, dt \, ds \end{aligned}$$

to obtain $g_y(x) = [\mathbf{D}f(0)^{-1}] \cdot (y + \varphi(x))$, $\|\varphi(x)\| \leq K\|x\|^2$ if $\|x\| \leq P$, and

$$\|g_y(x)\| \leq M(\|y\| + K\|x\|^2).$$

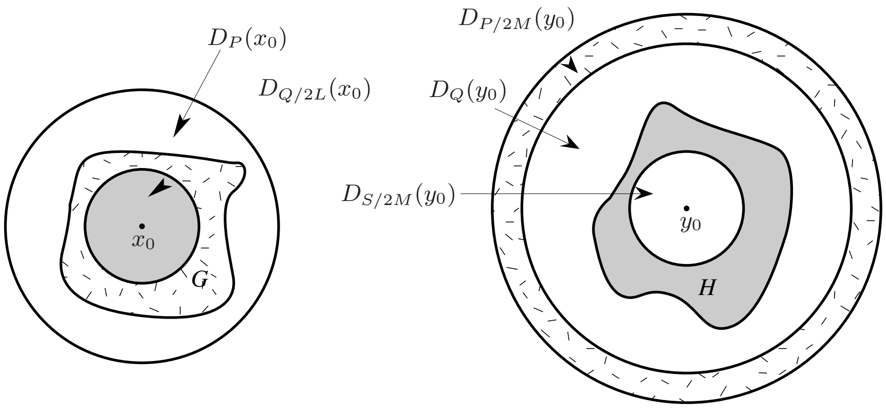

FIGURE 2.5.1. Regions for the proof of the inverse mapping theorem

Hence for $\|y\| \leq P/2M$, g_y maps $B_P(0)$ to $B_S(0)$. Similarly we get $\|g_y(x_1) - g_y(x_2)\| \leq \|x_1 - x_2\|/2$ from the mean value inequality and the estimate

$$\|\mathbf{D}g_y(x)\| = \|\mathbf{D}f(0)^{-1}\| \left(\left\| \int_0^1 \mathbf{D}^2 f(tx)x\,dt \right\| \right) \leq M(K\|x\|) \leq \frac{1}{2}$$

if $\|x\| \leq P$. Thus, as in the previous proof, $f^{-1} : B_{P/2M}(0) \to B_P(0)$ is defined and there exists an open set $G \subset B_P(0)$ diffeomorphic via f to the open ball $D_{P/2M}(0)$.

Taking the second derivative of the relation $f^{-1} \circ f =$ identity on G, we get

$$\mathbf{D}^2 f^{-1}(f(x))(\mathbf{D}f(x) \cdot u_1, \mathbf{D}f(x) \cdot u_2) + Df^{-1}(f(x)) \cdot \mathbf{D}^2 f(x)(u_1, u_2) = 0$$

for any $u_1, u_2 \in \mathbf{E}$. Let $v_i = \mathbf{D}f(x) \cdot u_i$, $i = 1, 2$, so that

$$\begin{aligned} \mathbf{D}^2 f^{-1}(f(x)) \cdot (v_1, v_2) = -\mathbf{D}f^{-1}(f(x)) \cdot \mathbf{D}^2 f(x)(&\mathbf{D}f(x)^{-1} \cdot v_1, \\ &\mathbf{D}f(x)^{-1} \cdot v_2) \end{aligned}$$

and hence

$$\begin{aligned} \left\|\mathbf{D}^2 f^{-1}(f(x))(v_1, v_2)\right\| &\leq \left\|\mathbf{D}f^{-1}(f(x))\right\|^3 \left\|\mathbf{D}^2 f(x)\right\| \|v_1\| \, \|v_2\| \\ &\leq 8M^3 K \|v_1\| \, \|v_2\| \end{aligned}$$

since $x \in G \subset D_P(0)$ and on $B_P(0)$ we have the inequality $\|\mathbf{D}f(x)^{-1}\| \leq 2M$. Thus on $B_{P/2M}(0)$ the following estimate holds:

$$\|\mathbf{D}^2 f^{-1}(y)\| \leq 8M^3 K.$$

By the previous argument with f replaced by f^{-1}, R by $P/2M$, L by M, and K by $N = 8M^3K$, it follows that there is an open set $H \subset D_Q(0)$, $Q = \min\{1/2KM, Q/2L, Q\}$ such that $f^{-1} : H \to D_{Q/2L}(0)$ is a diffeomorphism. Since f^{-1} is a diffeomorphism on $B_Q(0)$ and H is one of its open subsets, it follows that $B_{Q/2L}(0) \subset G$.

Finally, replacing R by $Q/2L$, we conclude the existence of a ball $B_{S/2M}(0)$, where $S = \min\{1/2KM, Q/2L, Q\}$, on which f^{-1} is a diffeomorphism. Therefore $B_{S/2M}(0) \subset H$. ■

Implicit Function Theorem. In the study of manifolds and submanifolds, the argument used in the following is of central importance.

2.5.7 Theorem (Implicit Function Theorem). *Let $U \subset \mathbf{E}$, $V \subset \mathbf{F}$ be open and $f : U \times V \to \mathbf{G}$ be C^r, $r \geq 1$. For some $x_0 \in U, y_0 \in V$ assume the partial derivative in the second slot $\mathbf{D}_2 f(x_0, y_0) : \mathbf{F} \to \mathbf{G}$, is an isomorphism. Then there are neighborhoods U_0 of x_0 and W_0 of $f(x_0, y_0)$ and a unique C^r map $g : U_0 \times W_0 \to V$ such that for all $(x, w) \in U_0 \times W_0$,*

$$f(x, g(x, w)) = w.$$

Proof. Define the map

$$\Phi : U \times V \to \mathbf{E} \times \mathbf{G}$$

by $(x, y) \mapsto (x, f(x, y))$. Then $\mathbf{D}\Phi(x_0, y_0)$ is given by

$$\mathbf{D}\Phi(x_0, y_0) \cdot (x_1, y_1) = \begin{pmatrix} I & 0 \\ \mathbf{D}_1 f(x_0, y_0) & \mathbf{D}_2 f(x_0, y_0) \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

which is an isomorphism of $\mathbf{E} \times \mathbf{F}$ with $\mathbf{E} \times \mathbf{G}$. Thus, Φ has a unique C^r local inverse, say $\Phi^{-1} : U_0 \times W_0 \to U \times V$, $(x, w) \mapsto (x, g(x, w))$. The g so defined is the desired map. ■

Applying the chain rule to the relation $f(x, g(x, w))$, one can compute the derivatives of g:

$$\mathbf{D}_1 g(x, w) = -[\mathbf{D}_2 f(x, g(x, w))]^{-1} \circ \mathbf{D}_1 f(x, g(x, w)),$$
$$\mathbf{D}_2 g(x, w) = [\mathbf{D}_2 f(x, g(x, w))]^{-1}.$$

2.5.8 Corollary. *Let $U \subset \mathbf{E}$ be open and $f : U \to \mathbf{F}$ be C^r, $r \geq 1$. Suppose $\mathbf{D}f(x_0)$ is surjective and* $\ker \mathbf{D}f(x_0)$ *is complemented. Then $f(U)$ contains a neighborhood of $f(x_0)$.*

Proof. Let $\mathbf{E}_1 = \ker \mathbf{D}f(x_0)$ and $\mathbf{E} = \mathbf{E}_1 \oplus \mathbf{E}_2$. Then $\mathbf{D}_2 f(x_0) : \mathbf{E}_2 \to \mathbf{F}$ is an isomorphism, so the hypotheses of Theorem 2.5.7 are satisfied and thus $f(U)$ contains W_0 provided by that theorem. ■

Local Surjectivity Theorem. Since in finite-dimensional spaces every subspace splits, the foregoing corollary implies that if $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$, $n \geq m$, and the Jacobian of f at every point of U has rank m, then f is an open mapping. This statement generalizes directly to Banach spaces, but it is not a consequence of the implicit function theorem anymore, since not every subspace is split. This result goes back to Graves [1950]. The proof given in Supplement 2.5B follows Luenberger [1969].

2.5.9 Theorem (Local Surjectivity Theorem). *If $f : U \subset \mathbf{E} \to \mathbf{F}$ is C^1 and $\mathbf{D}f(u_0)$ is onto for some $u_0 \in U$, then f is locally onto; that is, there exist open neighborhoods U_1 of u_0 and V_1 of $f(u_0)$ such that $f|U_1 : U_1 \to V_1$ is onto. In particular, if $\mathbf{D}f(u)$ is onto for all $u \in U$, then f is an open mapping.*

Supplement 2.5B
Proof of the Local Surjectivity Theorem

Proof. Recall from §2.1 that $\mathbf{E}/\ker \mathbf{D}f(u_0) = \mathbf{E}_0$ is a Banach space with norm $\|[x]\| = \inf\{\, \|x + u\| \mid u \in \ker \mathbf{D}f(u_0)\,\}$, where $[x]$ is the equivalence class of x. To solve $f(x) = y$ we set up an iteration scheme in $\mathbf{E}_0$ and $\mathbf{E}$ simultaneously. Now $\mathbf{D}f(u_0)$ induces an isomorphism $T : \mathbf{E}_0 \to \mathbf{F}$, so $T^{-1} \in L(\mathbf{F}, \mathbf{E}_0)$ exists by the Banach isomorphism theorem. Let $x = u_0 + h$ and write $f(x) = y$ as

$$T^{-1}(y - f(u_0 + h)) = 0.$$

To solve this equation, define a sequence $L_n \in \mathbf{E}/\ker \mathbf{D}f(u_0)$ (so the element L_n is a coset of $\ker \mathbf{D}f(u_0)$) and $h_n \in L_n \subset \mathbf{E}$ inductively by $L_0 = \ker \mathbf{D}f(u_0)$, $h_0 \in L_0$ small, and

$$L_n = L_{n-1} + T^{-1}(y - f(u_0 + h_{n-1})), \tag{2.5.1}$$

and selecting $h_n \in L_n$ such that

$$\|h_n - h_{n-1}\| \leq 2\|L_n - L_{n-1}\|. \tag{2.5.2}$$

The latter is possible since

$$\|L_n - L_{n-1}\| = \inf\{\, \|h - h_{n-1}\| \mid h \in L_n \,\}.$$

Since $h_{n-1} \in L_{n-1}$, $L_{n-1} = T^{-1}(\mathbf{D}f(u_0) \cdot h_{n-1})$, so

$$L_n = T^{-1}(y - f(u_0 + h_{n-1}) + \mathbf{D}f(u_0) \cdot h_{n-1}).$$

Subtracting this from the expression for L_{n-1} gives

$$\begin{aligned} L_n - L_{n-1} = -T^{-1}(&f(u_0 + h_{n-1}) - f(u_0 + h_{n-2}) \\ &- \mathbf{D}f(u_0) \cdot (h_{n-1} - h_{n-2})). \end{aligned}$$

For $\varepsilon > 0$ given, there is a neighborhood U of u_0 such that

$$\|\mathbf{D}f(u) - \mathbf{D}f(u_0)\| < \varepsilon$$

for $u \in U$, since f is C^1. Assume inductively that $u_0 + h_{n-1} \in U$ and $u_0 + h_{n-2} \in U$. Then from the mean value inequality,

$$\|L_n - L_{n-1}\| \leq \varepsilon \|T^{-1}\| \, \|h_{n-1} - h_{n-2}\|. \tag{2.5.3}$$

By equation (2.5.2),

$$\|h_n - h_{n-1}\| \leq 2\|L_n - L_{n-1}\| \leq 2\varepsilon \|T^{-1}\| \, \|h_{n-1} - h_{n-2}\|.$$

Thus if ε is small,

$$\|h_n - h_{n-1}\| \leq \frac{1}{2}\|h_{n-1} - h_{n-2}\|.$$

Starting with h_0 small and $\|h_1 - h_0\| < (1/2)\|h_0\|$, $u_0 + h_n$ remain inductively in U since

$$\begin{aligned} \|h_n\| &\leq \|h_0\| + \|h_1 - h_0\| + \|h_2 - h_1\| + \cdots + \|h_n - h_{n-1}\| \\ &\leq \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right) \|h_0\| \leq 2\|h_0\|. \end{aligned}$$

It follows that h_n is a Cauchy sequence, so it converges to some point, say h. Correspondingly, L_n converges to L and $h \in L$. Thus from equation (2.5.1), $0 = T^{-1}(y - f(u_0 + h))$ and so $y = f(u_0 + h)$. ■

The local surjectivity theorem shows that for y near $y_0 = f(u_0)$, $f(x) = y$ has a solution. If there is a solution $g(y) = x$ which is C^1, then $\mathbf{D}f(x_0) \circ \mathbf{D}g(y_0) = I$ and so range $\mathbf{D}g(y_0)$ is an algebraic complement to $\ker \mathbf{D}f(x_0)$. It follows that if range $\mathbf{D}g(y_0)$ is closed, then $\ker \mathbf{D}f(x_0)$ is split.

In many applications to nonlinear partial differential equations, methods of functional analysis and elliptic operators can be used to show that $\ker \mathbf{D}f(x_0)$ does split, even in Banach spaces. Such a splitting theorem is called the ***Fredholm alternative***. For illustrations of this idea in geometry and relativity, see Fischer and Marsden [1975, 1979], and in elasticity, see Chapter 6 of Marsden and Hughes [1983]. For such applications, Corollary 2.5.8 often suffices.

Local Injectivity Theorem. The locally injective counterpart of this theorem is the following.

2.5.10 Theorem (Local Injectivity Theorem)**.** *Let $f : U \subset \mathbf{E} \to \mathbf{F}$ be a C^1 map, $\mathbf{D}f(u_0)(\mathbf{E})$ be closed in $\mathbf{F}$, and $\mathbf{D}f(u_0) \in \mathrm{GL}(\mathbf{E}, \mathbf{D}f(u_0)(\mathbf{E}))$. Then there exists a neighborhood V of u_0, $V \subset U$, on which f is injective. The inverse $f^{-1} : f(V) \to U$ is Lipschitz continuous.*

Proof. Since $(\mathbf{D}f(u_0))^{-1} \in L(\mathbf{D}f(u_0)(\mathbf{E}), \mathbf{E})$, there is a constant $M > 0$ such that $\|\mathbf{D}f(u_0) \cdot e\| \geq M\|e\|$ for all $e \in \mathbf{E}$. By continuity of $\mathbf{D}f$, there exists $r > 0$ such that $\|\mathbf{D}f(u) - \mathbf{D}f(u_0)\| < M/2$ whenever $\|u - u_0\| < 3r$. By the mean value inequality, for $e_1, e_2 \in D_r(u_0)$

$$\begin{aligned} &\|f(e_1) - f(e_2) - \mathbf{D}f(u_0)(e_1 - e_2)\| \\ &\qquad \leq \sup_{t \in [0,1]} \|\mathbf{D}f(e_1 + t(e_2 - e_1)) - \mathbf{D}f(u_0)\| \, \|e_1 - e_2\| \\ &\qquad \leq \frac{M\|e_1 - e_2\|}{2} \end{aligned}$$

since $\|u_0 - e_1 - t(e_2 - e_1)\| < 3r$. Thus

$$M\|e_1 - e_2\| \leq \|\mathbf{D}f(u_0) \cdot (e_1 - e_2)\| \leq \|f(e_1) - f(e_2)\| + \frac{M}{2}\|e_1 - e_2\|;$$

that is,

$$\frac{M}{2}\|e_1 - e_2\| \leq \|f(e_1) - f(e_2)\|,$$

which proves that f is injective on $D_r(u_0)$ and that $f^{-1} : f(D_r(u_0)) \to U$ is Lipschitz continuous. ■

Notice that this proof is done by direct estimates, and not by invoking the inverse or implicit function theorem. If, however, the range space $\mathbf{D}f(u_0)(\mathbf{E})$ splits, one could alternatively prove results like this by composing f with the projection onto this range and applying the inverse function theorem to the composition. In the following paragraphs on local immersions and submersions, we examine this point of view in detail.

Application to Differential Equations. We now give an example of the use of the implicit function theorem to prove an existence theorem for differential equations. For this and related examples, we choose the spaces to be infinite dimensional. In fact, $\mathbf{E}, \mathbf{F}, \mathbf{G}, \cdots$ will be suitable spaces of functions. The map f will often be a nonlinear differential operator. The linear map $\mathbf{D}f(x_0)$ is called the ***linearization*** of f about x_0. (Phrases like "first variation," "first-order deformation," and so forth are also used.)

2.5.11 Example. Let $\mathbf{E}$ be the space of all C^1-functions $f : [0,1] \to \mathbb{R}$ with the norm

$$\|f\|_1 = \sup_{x \in [0,1]} |f(x)| + \sup_{x \in [0,1]} \left| \frac{df(x)}{dx} \right|$$

and $\mathbf{F}$ the space of all C^0-functions with the norm $\|f\|_0 = \sup_{x \in [0,1]} |f(x)|$. These are Banach spaces (see Exercise 2.1-3). Let $\Phi : \mathbf{E} \to \mathbf{F}$ be defined by $\Phi(f) = df/dx + f^3$. It is easy to check that Φ is C^∞ and $\mathbf{D}\Phi(0) = d/dx : \mathbf{E} \to \mathbf{F}$. Clearly $\mathbf{D}\Phi(0)$ is surjective (fundamental theorem of calculus). Also $\ker \mathbf{D}\Phi(0)$ consists of $\mathbf{E}_1 =$ all constant functions. This is complemented because it is finite dimensional; explicitly, a complement consists of functions with zero integral. Thus, Corollary 2.5.8 yields the following:

There is an $\varepsilon > 0$ such that if $g : [0,1] \to \mathbb{R}$ is a continuous function with $|g(x)| < \varepsilon$, then there is a C^1 function $f : [0,1] \to \mathbb{R}$ such that

$$\frac{df}{dx} + f^3(x) = g(x).$$

♦

Supplement 2.5C

An Application of the Inverse Function Theorem to a Nonlinear Partial Differential Equation

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Consider the problem

$$\nabla^2\varphi + \varphi^3 = f \quad \text{in } \Omega, \quad \varphi + \varphi^7 = g \quad \text{on } \partial\Omega$$

for given f and g. We claim that for f and g small, this problem has a unique small solution. For partial differential equations of this sort one can use the Sobolev spaces $H^s(\Omega, \mathbb{R})$ consisting of maps $\varphi : \Omega \to \mathbb{R}$ whose first s distributional derivatives lie in L^2. (One uses Fourier transforms to define this space if s is not an integer.) In the Sobolev spaces $\mathbf{E} = H^s(\Omega, \mathbb{R})$, $\mathbf{F} = H^{s-2}(\Omega, \mathbb{R}) \times H^{s-1/2}(\partial\Omega, \mathbb{R})$, if $s > n/2$ the map

$$\Phi : \mathbf{E} \to \mathbf{F}, \quad \varphi \mapsto (\nabla^2\varphi + \varphi^3, (\varphi + \varphi^7)|\partial\Omega)$$

is C^∞ (use Supplement 2.4B) and the linear operator

$$\mathbf{D}\Phi(0) \cdot \varphi = (\nabla^2\varphi, \varphi|\partial\Omega)$$

is an isomorphism. The fact that $\mathbf{D}\Phi(0)$ is an isomorphism is a result on the solvability of the Dirichlet problem from the theory of elliptic linear partial differential equations. See, for example, Friedman [1969]. (In the C^k spaces, $\mathbf{D}\Phi(0)$ is *not* an isomorphism.) The result claimed above now follows from the inverse function theorem.

Local Immersions and Submersions. The following series of consequences of the inverse function theorem are important technical tools in the study of manifolds. The first two results give, roughly speaking, sufficient conditions to "straighten out" the range (respectively, the domain) of f in a neighborhood of a point, thus making f look like an inclusion (respectively, a projection).

2.5.12 Theorem (Local Immersion Theorem). *Let $f : U \subset \mathbf{E} \to \mathbf{F}$ be of class C^r, $r \geq 1$, $u_0 \in U$ and suppose that $\mathbf{D}f(u_0)$ is one-to-one and has a closed split image $\mathbf{F}_1$ with closed complement $\mathbf{F}_2$. (If $\mathbf{E} = \mathbb{R}^m$ and $\mathbf{F} = \mathbb{R}^n$, assume only that $\mathbf{D}f(u_0)$ has trivial kernel.) Then there are two open sets $U' \subset \mathbf{F}$ and $V \subset \mathbf{E} \oplus \mathbf{F}_2$, where $f(u_0) \in U'$ and a C^r diffeomorphism $\varphi : U' \to V$ such that $(\varphi \circ f)(e) = (e, 0)$ for all $e \in V \cap (\mathbf{E} \times \{0\}) \subset \mathbf{E}$.*

The intuition for $\mathbf{E} = \mathbf{F}_1 = \mathbb{R}^2$, $\mathbf{F}_2 = \mathbb{R}$ (i.e., $m = 2$, $n = 3$) is given in Figure 2.5.2. The function φ flattens out the image of f. Notice that this is intuitively correct; we expect the range of f to be an m-dimensional "surface" so it should be possible to flatten it to a piece of $\mathbb{R}^m$. Note that the range of a linear map of rank m is a linear subspace of dimension exactly m, so this result expresses, in a sense, a generalization of the linear case. Also note that Theorem 2.5.10, the local injectivity theorem, follows from the more restrictive hypotheses of Theorem 2.5.12.

Proof. Define $g : U \times \mathbf{F}_2 \subset \mathbf{E} \times \mathbf{F}_2 \to \mathbf{F} = \mathbf{F}_1 \oplus \mathbf{F}_2$ by $g(u, v) = f(u) + (0, v)$ and note that $g(u, 0) = f(u)$. Now

$$\mathbf{D}g(u_0, 0) = (\mathbf{D}f(u_0), I_{\mathbf{F}_2}) \in \mathrm{GL}(\mathbf{E} \oplus \mathbf{F}_2, \mathbf{F})$$

by the Banach isomorphism theorem. Here, $I_{\mathbf{F}_2}$ denotes the identity mapping of $\mathbf{F}_2$ and for $A \in L(\mathbf{E}, \mathbf{F})$ and $B \in L(\mathbf{E}', \mathbf{F}')$, the element $(A, B) \in L(\mathbf{E} \oplus \mathbf{E}', \mathbf{F} \oplus \mathbf{F}')$ is defined by $(A, B)(e, e') = (Ae, Be')$. By the inverse function theorem there exist open sets U' and V such that $(u_0, 0) \in V \subset \mathbf{E} \oplus \mathbf{F}_2$, and $g(u_0, 0) = f(u_0) \in U' \subset \mathbf{F}$ and a C^r diffeomorphism $\varphi : U' \to V$ such that $\varphi^{-1} = g|V$. Hence for $(e, 0) \in V$, $(\varphi \circ f)(e) = (\varphi \circ g)(e, 0) = (e, 0)$. ∎

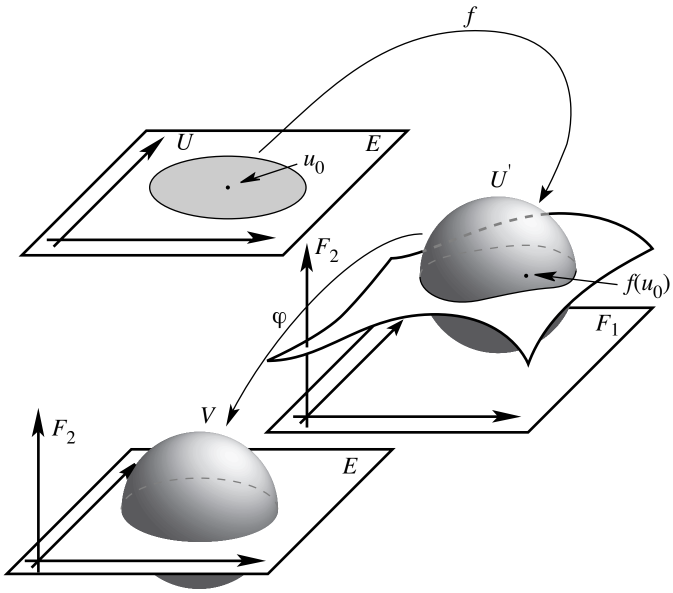

FIGURE 2.5.2. The local immersion theorem

2.5.13 Theorem (Local Submersion Theorem). *Let $f : U \subset \mathbf{E} \to \mathbf{F}$ be of class C^r, $r \geq 1$, $u_0 \in U$ and suppose $\mathbf{D}f(u_0)$ is surjective and has split kernel $\mathbf{E}_2$ with closed complement $\mathbf{E}_1$. (If $\mathbf{E} = \mathbb{R}^m$ and $\mathbf{F} = \mathbb{R}^n$, assume only that rank $(\mathbf{D}f(u_0)) = n$.) Then there are open sets U' and V such that $u_0 \in U' \subset U \subset \mathbf{E}$ and $V \subset \mathbf{F} \oplus \mathbf{E}_2$ and a C^r diffeomorphism $\psi : V \to U'$ with the property that $(f \circ \psi)(u, v) = u$ for all $(u, v) \in V$.*

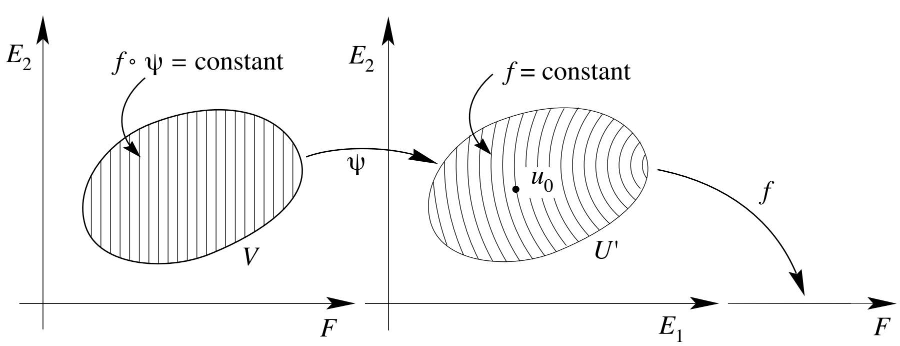

FIGURE 2.5.3. A submersion theorem

The intuition for the special case $\mathbf{E}_1 = \mathbf{E}_2 = \mathbf{F} = \mathbb{R}$ is given in Figure 2.5.3, which should be compared to Figure 2.5.2. Note that this theorem implies the results of Theorem 2.5.9, the local surjectivity theorem, but the hypotheses are more stringent.

Proof. By the Banach isomorphism theorem (§2.2), $\mathbf{D}_1 f(u_0) \in \mathrm{GL}(\mathbf{E}_1, \mathbf{F})$. Define the map

$$g : U \subset \mathbf{E}_1 \oplus \mathbf{E}_2 \to \mathbf{F} \oplus \mathbf{E}_2$$

by $g(u_1, u_2) = (f(u_1, u_2), u_2)$ and note that

$$\mathbf{D}g(u_0) \cdot (e_1, e_2) = \begin{bmatrix} \mathbf{D}_1 f(u_0) & \mathbf{D}_2 f(u_0) \\ 0 & I_{\mathbf{E}_2} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

so that $\mathbf{D}g(u_0) \in \mathrm{GL}(\mathbf{E}, \mathbf{F} \oplus \mathbf{E}_2)$. By the inverse function theorem there are open sets U' and V such that $u_0 \in U' \subset U \subset \mathbf{E}$, $V \subset \mathbf{F} \oplus \mathbf{E}_2$ and a C^r diffeomorphism $\psi : V \to U'$ such that $\psi^{-1} = g|U'$. Hence if $(u, v) \in V$,

$$(u, v) = (g \circ \psi)(u, v) = (f(\psi(u, v)), \psi_2(u, v)),$$

where $\psi = \psi_1 \times \psi_2$; that is, $\psi_2(u, v) = v$ and $(f \circ \psi)(u, v) = u$. ■

Local Representation and Rank Theorems. We now give two results that extend the above results on the local structure of maps.

2.5.14 Theorem (Local Representation Theorem). *Let $f : U \subset \mathbf{E} \to \mathbf{F}$ be of class C^r, $r \geq 1$, $u_0 \in U$ and suppose $\mathbf{D}f(u_0)$ has closed split image $\mathbf{F}_1$ with closed complement $\mathbf{F}_2$ and split kernel $\mathbf{E}_2$ with closed complement $\mathbf{E}_1$. (If $\mathbf{E} = \mathbb{R}^m$, $\mathbf{F} = \mathbb{R}^n$, assume that* $\mathrm{rank}(\mathbf{D}f(u_0)) = k$*, $k \leq n$, $k \leq m$, so that $\mathbf{F}_2 = \mathbb{R}^{n-k}$, $\mathbf{F}_1 = \mathbb{R}^k$, $\mathbf{E}_1 = \mathbb{R}^k$, $\mathbf{E}_2 = \mathbb{R}^{m-k}$.) Then there are open sets U' and V with the property that $u_0 \in U' \subset U \subset \mathbf{E}$, $V \subset \mathbf{F}_1 \oplus \mathbf{E}_2$ and a C^r diffeomorphism $\psi : V \to U'$ such that $(f \circ \psi)(u, v) = (u, \eta(u, v))$, where $\eta : V \to \mathbf{F}_2$ is a C^r map satisfying $\mathbf{D}\eta\left(\psi^{-1}(u_0)\right) = 0$.*

Proof. Write $f = f_1 \times f_2$, where $f_i : U \to \mathbf{F}_i$, $i = 1, 2$. Then f_1 satisfies the conditions of Theorem 2.5.13, and thus there exists a C^r diffeomorphism $\psi : V \subset \mathbf{F}_1 \oplus \mathbf{E}_2 \to U' \subset \mathbf{E}$ such that the composition $f_1 \circ \psi$ is given by $(f_1 \circ \psi)(u, v) = u$. Let $\eta = f_2 \circ \psi$. ■

To use Theorem 2.5.12 (or Theorem 2.5.13) in finite dimensions, we must have the rank of $\mathbf{D}f$ equal to the dimension of its domain space (or the range space). However, we can also use the inverse function theorem to tell us that if $\mathbf{D}f(x)$ has constant rank k in a neighborhood of x_0, then we can straighten out the domain of f with some invertible function ψ such that $f \circ \psi$ depends only on k variables. Then we can apply the local immersion theorem (Theorem 2.5.12). This is the essence of the following theorem.

Roughly speaking, in finite dimensions, the rank theorem says that if $\mathbf{D}f$ has constant rank k on an open set in $\mathbb{R}^m$, then $m - k$ variables are redundant and can be eliminated. As a simple example, if $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by setting $f(x, y) = x - y$, then $\mathbf{D}f$ has rank 1, and indeed, we can express f using just one variable, namely, let $\psi(x, y) = (x + y, y)$ so that $(f \circ \psi)(x, y) = x$, which depends only on x.

2.5.15 Theorem (Rank Theorem). *Let $f : U \subset \mathbf{E} \to \mathbf{F}$ be of class C^r, $r \geq 1$, $u_0 \in U$ and suppose $\mathbf{D}f(u_0)$ has closed split image $\mathbf{F}_1$ with closed complement $\mathbf{F}_2$ and split kernel $\mathbf{E}_2$ with closed complement $\mathbf{E}_1$. In addition, assume that for all u in a neighborhood of $u_0 \in U$, $\mathbf{D}f(u)(\mathbf{E})$ is a closed subspace of $\mathbf{F}$ and $\mathbf{D}f(u)|\mathbf{E}_1 : \mathbf{E}_1 \to \mathbf{D}f(u)(\mathbf{E})$ is a Banach space isomorphism. (In case $\mathbf{E} = \mathbb{R}^m$ and $\mathbf{F} = \mathbb{R}^n$, assume only that* $\mathrm{rank}(\mathbf{D}f(u)) = k$ *for u in a neighborhood of u_0.) Then there exist open sets*

$$U_1 \subset \mathbf{F}_1 \oplus \mathbf{E}_2, \quad U_2 \subset \mathbf{E}, \quad V_1 \subset \mathbf{F}, \quad \textit{and} \quad V_2 \subset \mathbf{F}$$

and there are C^r diffeomorphisms $\varphi : V_1 \to V_2$ and $\psi : U_1 \to U_2$ such that $(\varphi \circ f \circ \psi)(x, e) = (x, 0)$.

The intuition is given by Figure 2.5.4 for $\mathbf{E} = \mathbb{R}^2$, $\mathbf{F} = \mathbb{R}^2$, and $k = 1$.

Remark. It is clear that the theorem implies $\mathbf{E}_1 \oplus \ker(\mathbf{D}f(u)) = \mathbf{E}$ and $\mathbf{D}f(u)(\mathbf{E}) \oplus \mathbf{F}_2 = \mathbf{F}$ for u in a neighborhood of u_0 in U, because $\varphi \circ f \circ \psi$ has these properties. These seemingly stronger conditions can in fact be shown directly to be equivalent to the hypotheses in the theorem by the use of the openness of $\mathrm{GL}(\mathbf{E}, \mathbf{E})$ in $L(\mathbf{E}, \mathbf{E})$. ♦

Proof. By the local representation theorem there is a C^r diffeomorphism $\psi : U_1 \subset \mathbf{F}_1 \oplus \mathbf{E}_2 \to U_2 \subset \mathbf{E}$ such that $f(x, y) := (f \circ \psi)(x, y) = (x, \eta(x, y))$. Let $P_1 : \mathbf{F} \to \mathbf{F}_1$ be the projection. Since

$$\mathbf{D}f(x, y) \cdot (w, e) = (w, \mathbf{D}\eta(x, y) \cdot (w, e)),$$

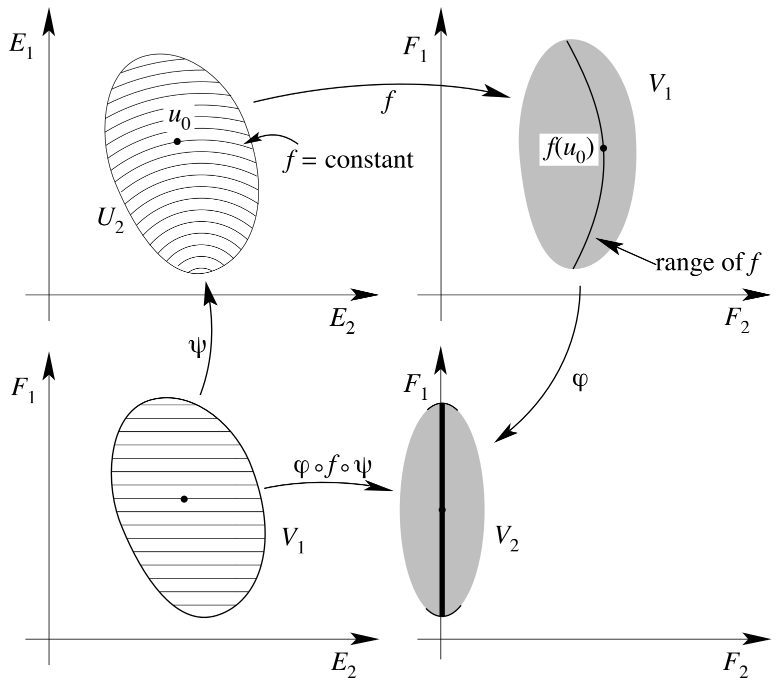

FIGURE 2.5.4. The rank theorem

it follows that $P_1 \circ \mathbf{D}f(x,y)(w,e) = (w,0)$, for $w \in \mathbf{F}_1$ and $e \in \mathbf{E}_2$. In particular $P_1 \circ \mathbf{D}f(x,y)|\mathbf{F}_1 \times \{0\} = I_1$, the identity on $\mathbf{F}_1$, which shows that

$$\mathbf{D}f(x,y)|\mathbf{F}_1 \times \{0\} : \mathbf{F}_1 \times \{0\} \to \mathbf{D}f(x,y)(\mathbf{F}_1 \oplus \mathbf{E}_2)$$

is injective. In finite dimensions this implies that it is an isomorphism, since $\dim(\mathbf{F}_1) = \dim(\mathbf{D}f(x,y)(\mathbf{F}_1 \oplus \mathbf{E}_2))$. In infinite dimensions this is our hypothesis. Thus, we get

$$\mathbf{D}f(x,y) \circ P_1|\mathbf{D}f(x,y)(\mathbf{F}_1 \oplus \mathbf{E}_2) = \text{identity}.$$

Let $(w, \mathbf{D}\eta(x,y)(w,e)) \in \mathbf{D}f(x,y)(\mathbf{F}_1 \oplus \mathbf{E}_2)$. Since

$$\begin{aligned}(\mathbf{D}f(x,y) \circ P_1)(w, \mathbf{D}\eta(x,y) \cdot (w,e)) &= \mathbf{D}f(x,y) \cdot (w,0) \\ &= (w, \mathbf{D}\eta(x,y) \cdot (w,0)) \\ &= (w, \mathbf{D}_1\eta(x,y) \cdot w),\end{aligned}$$

we must have $\mathbf{D}\eta(x,y) \cdot e = 0$ for all $e \in \mathbf{E}_2$; that is, $D_2\eta(x,y) = 0$. However, $\mathbf{D}^2 f(x,y) \cdot e = (0, \mathbf{D}_2\eta(x,y) \cdot e)$, which says that $\mathbf{D}^2 f(x,y) = 0$; that is, f does not depend on the variable $y \in \mathbf{E}_2$. Define

$$f(x) = f(x,y) = (f \circ y)(x,y),$$

so $f : P_1'(V) \subset \mathbf{F}_1 \to \mathbf{F}$ where $P_1' : \mathbf{F}_1 \oplus \mathbf{E}_2 \to \mathbf{F}_1$ is the projection. Now f satisfies the conditions of Theorem 2.5.12 at $P_1'(\psi^{-1}(u_0))$ and hence there exists a C^r diffeomorphism $\varphi : V_1 \to V_2$, where $V_1, V_2 \subset \mathbf{F}$, such that $(\varphi \circ f)(z) = (z,0)$; that is, we have $(\varphi \circ f \circ \psi)(x,y) = (x,0)$. ■

2.5.16 Example (Functional Dependence). Let $U \subset \mathbb{R}^n$ be an open set and let the functions $f_1, \ldots, f_n : U \to \mathbb{R}$ be smooth. The functions $f_1, \ldots, f_n$ are said to be ***functionally dependent*** at $x_0 \in U$ if there is a neighborhood V of the point $(f_1(x_0), \ldots, f_n(x_0)) \in \mathbb{R}^n$ and a smooth function $F : V \to \mathbb{R}$ such that $\mathbf{D}F \neq 0$ on a neighborhood of $(f_1(x_0), \ldots, f_n(x_0))$, and

$$F(f_1(x), \ldots, f_n(x)) = 0$$

for all x in some neighborhood of x_0. Show:

(i) *If $f = (f_1, \cdots, f_n)$ and $f_1, \cdots, f_n$ are functionally dependent at x_0, then the determinant of $\mathbf{D}f$, denoted*

$$Jf = \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_1)},$$

vanishes at x_0.

(ii) *If*

$$\frac{\partial(f_1, \ldots, f_{n-1})}{\partial(x_1, \ldots, x_{n-1})} \neq 0 \quad \text{and} \quad \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)} = 0$$

on a neighborhood of x_0, then $f_1, \ldots, f_n$ are functionally dependent, and $f_n = G(f_1, \ldots, f_{n-1})$ for some G.

Solution. (i) We have $F \circ f = 0$, so

$$\mathbf{D}F(f(x)) \circ \mathbf{D}f(x) = 0.$$

Now if $Jf(x_0) \neq 0$, $\mathbf{D}f(x)$ would be invertible in a neighborhood of x_0, implying $\mathbf{D}F(f(x)) = 0$. By the inverse function theorem, this implies $\mathbf{D}F(y) = 0$ on a whole neighborhood of $f(x_0)$.

(ii) The conditions of (ii) imply that $\mathbf{D}f$ has rank $n-1$. Hence by the rank theorem, there are mappings φ and ψ such that

$$(\varphi \circ f \circ \psi)(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, 0).$$

Let F be the last component of φ . Then $F(f_1, \ldots, f_n) = 0$. Since φ is invertible, $\mathbf{D}F \neq 0$.

It follows from the implicit function theorem that we can locally solve $F(f_1, \ldots, f_n) = 0$ for $f_n = G(f_1, \ldots, f_{n-1})$, provided we can show $\Delta = \partial F/\partial y_n \neq 0$. As we saw before, $\mathbf{D}F(f(x)) \circ \mathbf{D}f(x) = 0$, or, in components with $y = f(x)$,

$$\left(\frac{\partial F}{\partial y_1} \cdots \frac{\partial F}{\partial y_n}\right) \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix} = (0, 0, \ldots, 0).$$

If $\partial F/\partial y_n = 0$, we would have

$$\left(\frac{\partial F}{\partial y_1}, \ldots, \frac{\partial F}{\partial y_{n-1}}\right) \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_{n-1}} \\ \vdots & & \vdots \\ \dfrac{\partial f_{n-1}}{\partial x_1} & \cdots & \dfrac{\partial f_{n-1}}{\partial x_{n-1}} \end{bmatrix} = (0, 0, \ldots, 0)$$

i.e.,

$$\left(\frac{\partial F}{\partial y_1}, \ldots, \frac{\partial F}{\partial y_{n-1}}\right) = (0, 0, \ldots, 0)$$

since the square matrix is invertible by the assumption that

$$\frac{\partial(f_1, \ldots, f_{n-1})}{\partial(x_1, \ldots, x_{n-1})} \neq 0.$$

This implies $\mathbf{D}f = 0$, which is not true. Hence $\partial F/\partial y_n \neq 0$, and we have the desired result. ♦

Note the analogy between linear dependence and functional dependence, where rank or determinant conditions are replaced by the analogous conditions on the Jacobian matrix.

Supplement 2.5D
The Hadamard–Levy Theorem

This supplement gives sufficient conditions which together with the hypotheses of the inverse function theorem guarantee that a C^k map f between Banach spaces is a global diffeomorphism. To get a feel for these supplementary conditions, consider a C^k function $f : \mathbb{R} \to \mathbb{R}$, $k \geq 1$, satisfying $1/|f'(x)| < M$ for all $x \in \mathbb{R}$. Then f is a local diffeomorphism at every point of $\mathbb{R}$ and thus is an open map. In particular, $f(\mathbb{R})$ is an open interval $]a, b[$. The condition $|f'(x)| > 1/M$ implies that f is either strictly increasing or strictly decreasing. Let us assume that f is strictly increasing. If $b < +\infty$, then the line $y = b$ is a horizontal asymptote of the graph of f and therefore we should have $\lim_{x\to\infty} f'(x) = 0$ contradicting $|f'(x)| > 1/M$. One similarly shows that $a = -\infty$ and the same proof works if $f'(x) < -1/M$. The theorem below generalizes this result to the case of Banach spaces.

2.5.17 Theorem (The Hadamard–Levy Theorem)**.** *Let $f : \mathbf{E} \to \mathbf{F}$ be a C^k map of Banach spaces, $k \geq 1$. If $\mathbf{D}f(x)$ is an isomorphism of $\mathbf{E}$ with $\mathbf{F}$ for every $x \in \mathbf{E}$ and if there is a constant $M > 0$ such that $\|\mathbf{D}f(x)^{-1}\| < M$ for all $x \in \mathbf{E}$, then f is a diffeomorphism.*

The key to the proof of the theorem consists of a homotopy lifting argument. If X is a topological space, a continuous map $\varphi : X \to \mathbf{F}$ is said to ***lift to* E *through*** f, if there is a continuous map $\psi : X \to \mathbf{E}$ satisfying $f \circ \psi = \varphi$.

2.5.18 Lemma. *Let X be a connected topological space, $\varphi : X \to \mathbf{F}$ a continuous map and let $f : \mathbf{E} \to \mathbf{F}$ be a C^1 map with $\mathbf{D}f(x)$ an isomorphism for every $x \in \mathbf{E}$. Fix $u_0 \in \mathbf{E}$, $v_0 \in \mathbf{F}$, and $x_0 \in X$ satisfying $f(u_0) = v_0$ and $\varphi(x_0) = v_0$. Then if a lift ψ of φ through f with $\varphi(x_0) = u_0$ exists, it is unique.*

Proof. Let ψ' be another lift and define the sets

$$X_1 = \{\, x \in X \mid \varphi(x) = \psi'(x) \,\} \quad \text{and} \quad X_2 = \{\, x \in X \mid \psi(x) \neq \psi'(x) \,\},$$

so that $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \varnothing$. We shall prove that both X_1, X_2 are open. Since $x_0 \in X_1$, connectedness of X implies $X_2 = \varnothing$ and the lemma will be proved.

If $x \in X_1$, let U be an open neighborhood of $\psi(x) = \psi'(x)$ on which f is a diffeomorphism. Then $\psi^{-1}(U) \cap \varphi'^{-1}(U)$ is an open neighborhood of x contained in X_1.

If $x \in X_2$, let U (resp., U') be an open neighborhood of the point $\psi(x)$ (resp. of $\psi'(x)$) on which f is a diffeomorphism and such that $U \cap U' = \varnothing$. Then the set $\psi^{-1}(U) \cap \psi'^{-1}(U')$ is an open neighborhood of x contained in X_2. ■

A path $\gamma : [0,1] \to \mathbf{G}$, where $\mathbf{G}$ is a Banach space, is called C^1 if $\gamma|]0,1[$ is uniformly C^1 and the extension by continuity of γ' to $[0,1]$ has the values $\gamma'(0), \gamma'(1)$ equal to

$$\gamma'(0) = \lim_{h \downarrow 0} \frac{\gamma(h) - \gamma(0)}{h}, \quad \gamma'(1) = \lim_{h \downarrow 0} \frac{\gamma(1) - \gamma(1-h)}{h}.$$

2.5.19 Lemma (Homotopy Lifting Lemma)**.** *Under the hypotheses of Theorem* 2.5.17*, let* $H(t,s)$ *be a continuous map of* $[0,1] \times [0,1]$ *into* $\mathbf{F}$ *such that for each fixed* $s \in [0,1]$ *the path* $t \mapsto H(t,s)$ *is* C^1*. In addition, assume that* H *fixes endpoints, that is,* $H(0,s) = y_0$ *and* $H(1,s) = y_1$*, for all* $s \in [0,1]$*. If* $y_0 = f(x_0)$ *for some* $x_0 \in \mathbf{E}$*, there exists a unique lift* K *of* H *through* f *which is* C^1 *in* t *for every* s*. See Figure* 2.5.5*.*

Proof. Uniqueness follows by Lemma 2.5.18. By the inverse function theorem, there are open neighborhoods U of x_0 and V of y_0 such that $f|U : U \to V$ is a diffeomorphism. Since the open set $H^{-1}(U)$ contains the closed set $\{0\} \times [0,1]$, there exists $\varepsilon > 0$ such that $[0,\varepsilon[\times [0,1] \subset H^{-1}(U)$. Let $K : [0,\varepsilon[\times [0,1] \to \mathbf{E}$ be given by $K = f^{-1} \circ H$. Consider the set $A = \{\, \delta \in [0,1] \mid H : [0,\delta[\times [0,1] \to \mathbf{F}$ can be lifted through f to $\mathbf{E} \,\}$ which contains the interval $[0,\varepsilon[$. If $\alpha = \sup A$ we shall show first that $\alpha \in A$ and second that $\alpha = 1$. This will prove the existence of the lifting K.

$$\begin{tikzcd} & \mathbf{E} \\ {[0,1] \times [0,1]} \arrow[ur, "K"] \arrow[r, "H"'] & \mathbf{F} \arrow[u, "f"'] \end{tikzcd}$$

FIGURE 2.5.5. The homotopy lifting lemma

To show that $\alpha \in A$, note that for $0 \le t < \alpha$ we have $f \circ K = H$ and thus $\mathbf{D}f(K(1,s)) \circ \partial K/t = \partial H/\partial t$, which implies that

$$\left\| \frac{\partial K}{\partial t} \right\| \le M \sup_{t,s \in [0,1]} \left\| \frac{\partial H}{\partial t} \right\| = N.$$

Thus by the mean value inequality, if $\{t_n\}$ is an increasing sequence in A converging to a,

$$\|K(t_n, s) - K(t_m, s)\| \le N|t_n - t_m|,$$

which shows that $\{K(t_n, s)\}$ is a Cauchy sequence in $\mathbf{E}$, uniformly in $s \in [0,1]$. Let

$$K(\alpha, s) = \lim_{t_n \uparrow \alpha} K(t_n, s).$$

By continuity of f and H we have

$$f(K(\alpha, s)) = \lim_{t_n \uparrow \alpha} f(K(t_n, s)) = \lim_{t_n \uparrow \alpha} H(t_n, s) = H(\alpha, s),$$

which proves that $\alpha \in A$.

Next we show that $\alpha = 1$. If $\alpha < 1$ consider the curves $s \mapsto K(\alpha, s)$ and $s \mapsto H(\alpha, s) = f(K(\alpha, s))$. For each $s \in [0,1]$ choose open neighborhoods U_s of $K(\alpha, s)$ and V_s of $H(\alpha, s)$ such that $f|U_s : U_s \to V_s$ is a diffeomorphism. By compactness of the path $K(\alpha, s)$ in s, that is, of the set $\{\, K(\alpha, s) \mid s \in [0,1] \,\}$, finitely many of the U_s, say $U_1, \ldots, U_n$, cover it. Therefore the corresponding $V_1, \ldots, V_n$ cover $\{ H(\alpha, s) \mid s \in [0,1] \}$. Since $H^{-1}(V_i)$ contains the point (α, s_i), there exists $\varepsilon > 0$ such that

$$]\alpha - \varepsilon_i, \alpha + \varepsilon_i[\times]s_i - \delta_i, s_i + \delta_i[\subset H^{-1}(V_i),$$

where $]s_i - \delta_i, s_i + \delta_i[\; = H(\alpha, \cdot)^{-1}(V_i)$ and in particular $]s_i - \delta_i, s_i + \delta_i[$, $i = 1, \ldots, n$ cover $[0,1]$. Let $\varepsilon = \min\{e_1, \ldots, e_n\}$ and define $K : [0, \alpha + \varepsilon[\times [0,1] \to \mathbf{E}$ by

$$K(t,s) = \begin{cases} K(t,s), & \text{if } (t,s) \in [0, \alpha[\times \{(0,1)\}; \\ (f|U_i)^{-1}(H(t,s)), & \text{if } (t,s) \in [\alpha, \alpha + \varepsilon[\times]s_i - \delta_i, s_i + \delta_i[, \end{cases}$$

where $i = 1, \ldots, n$. By Lemma 2.5.18, K is a lifting of H, contradicting the definition of α.

Finally, K is C^1 in t for each s by the chain rule:

$$\frac{\partial K}{\partial t} = \mathbf{D}f(K(t,s))^{-1} \circ \frac{\partial H}{\partial t}. \qquad \blacksquare$$

Proof of Theorem 2.5.17. Let $y_0, y \in \mathbf{F}$ and consider the path $\gamma(t) = (1-t)y_0 + ty$. Regarding γ as defined on $[0,1] \times [0,1]$, independent of the second variable, the homotopy lifting lemma guarantees the existence of a C^1 path $\delta : [0,1] \to \mathbf{E}$ lifting γ, that is, $f \circ \delta = \gamma$. In particular, $f(\delta(1)) = \gamma(1) = y$ and thus f is surjective.

To show that f is injective, assume $x_1 \neq x_2$, $f(x_1) = f(x_2)$, and consider the path $\delta(t) = (1-t)x_1 + tx_2$. Then $\gamma(t) = f(\delta(t))$. By the homotopy lifting lemma, there exists a lift K of H through f. From $f \circ K = H$ it follows that the continuous curve $s \mapsto K(0,s)$ is mapped by f to the point $f(x_1)$, thus contradicting the inverse function theorem.

Therefore f is a bijective map which is a local diffeomorphism around every point, that is, f is a diffeomorphism of $\mathbf{E}$ with $\mathbf{F}$. $\blacksquare$

Remarks. (i) The uniform bound on $\|\mathbf{D}f(x)^{-1}\|$ can be replaced by properness of the map, that is, if $f(x_n) \to y$ there exists a convergent subsequence $\{x_m\}$, $x_m \to x$ with $f(x) = y$ (see Exercise 1.5-10). Indeed, the only place where the uniform bound on $\|\mathbf{D}f(x)^{-1}\|$ was used is in the homotopy lifting lemma in the argument that $\alpha = \sup A \in A$. If f is proper, this is shown in the following way. Let $\{t(n)\}$ be an increasing sequence in A converging to α. Then $H(t(n), s) \to H(\alpha, s)$ and from $f \circ K = H$ on $[0, \alpha[\times [0,1]$, it follows that $f(K(t(m), s)) \to H(\alpha, s)$ uniformly in $s \in [0,1]$. Thus, by properness of f, there is a subsequence $\{t(m)\}$ such that $K(t(m), s)$ is convergent for every s. Put $K(\alpha, s) = \lim_{t(m)\uparrow\alpha} K(t(n), s)$ and proceed as before.

(ii) If $\mathbf{E}$ and $\mathbf{F}$ are finite dimensional, properness of f is equivalent to: the inverse image of every compact set in $\mathbf{F}$ is compact in $\mathbf{E}$ (see Exercise 1.5-10).

(iii) Conditions on f like the one in (ii) or in the theorem are necessary as the following counterexample shows. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by (e^x, ye^{-x}) so that $f(\mathbb{R}^2)$ is the right open half plane and in particular f is not onto. However

$$\mathbf{D}f(x,y) = \begin{bmatrix} e^x & 0 \\ -ye^{-x} & e^{-x} \end{bmatrix}$$

is clearly an isomorphism for every $(x,y) \in \mathbb{R}^2$. But f is neither proper nor does the norm $\|\mathbf{D}f(x,y)^{-1}\|$ have a uniform bound on $\mathbb{R}^2$. For example, the inverse image of the compact set $[0,1] \times \{0\}$ is $]-\infty, 0] \times \{0\}$ and $\|\mathbf{D}f(x,y)^{-1}\| = C[e^{-2x} + e^{2x} + y^2 e^{-2x}]^{1/2}$, which is unbounded $x \to +\infty$.

(iv) See Wu and Desoer [1972] and Ichiraku [1985] for useful references to the theorem and applications. ♦

Lax–Milgram Theorem. If $\mathbf{E} = \mathbf{F} = \mathbf{H}$ is a Hilbert space, then the Hadamard–Levy theorem has an important consequence. We have seen that in the case of $f : \mathbb{R} \to \mathbb{R}$ with a uniform bound on $1/|f'(x)|$, the strong monotonicity of f played a key role in the proof that f is a diffeomorphism.

2.5.20 Definition. *Let $\mathbf{H}$ be a Hilbert space. A map $f : \mathbf{H} \to \mathbf{H}$ is strongly monotone if there exists $a > 0$ such that*

$$\langle f(x) - f(y), x - y\rangle \geq a\|x - y\|^2.$$

As in calculus, for differentiable maps strong monotonicity takes on a familiar form.

2.5.21 Lemma. *Let $f : \mathbf{H} \to \mathbf{H}$ be a differentiable map of the Hilbert space $\mathbf{H}$ onto itself. Then f is **strongly monotone** if and only if*

$$\langle \mathbf{D}f(x) \cdot u, u\rangle \geq a\|u\|^2$$

for some $a > 0$.

Proof. If f is strongly monotone, $\langle f(x + tu) - f(x), tu\rangle \geq at^2\|u\|^2$ for any $x, u \in \mathbf{H}$, $t \in \mathbb{R}$. Dividing by t^2 and taking the limit as $t \to 0$ yields the result.

Conversely, integrating both sides of $\langle \mathbf{D}f(x + tu) \cdot u, u\rangle \geq a\|u\|^2$ from 0 to 1 gives the strong monotonicity condition. ■

2.5.22 Lemma (Lax–Milgram Lemma)**.** *Let $\mathbf{H}$ be a real Hilbert space and $A \in L(\mathbf{H}, \mathbf{H})$ satisfy the estimate $\langle Ae, e\rangle \geq a\|e\|^2$ for all $e \in \mathbf{H}$. Then A is an isomorphism and $\|A^{-1}\| \leq 1/a$.*

Proof. The condition clearly implies injectivity of A. To prove A is surjective, we show first that $A(\mathbf{H})$ is closed and then that the orthogonal complement $A(\mathbf{H})^\perp$ is $\{0\}$. Let $f_n = A(e_n)$ be a sequence which converges to $f \in \mathbf{H}$. Since $\|Ae\| \geq a\|e\|$ by the Schwarz inequality, we have

$$\|f_n - f_m\| = \|A(e_n - e_m)\| \geq a\|e_n - e_m\|,$$

and thus $\{e_n\}$ is a Cauchy sequence in $\mathbf{H}$. If e is its limit we have $Ae = f$ and thus $f \in A(\mathbf{H})$.

To prove $A(\mathbf{H})^\perp = \{0\}$, let $u \in A(\mathbf{H})^\perp$ so that $0 = \langle Au, u\rangle \geq a\|u\|^2$ whence $u = 0$.

By Banach's isomorphism theorem 2.2.16, A is a Banach space isomorphism of $\mathbf{H}$ with itself. Finally, replacing e by $A^{-1}f$ in $\|Ae\| \geq a\|e\|$ yields $\|A^{-1}f\| \leq \|f\|/a$, that is, $\|A^{-1}\| \leq 1/a$. ■

Lemmas 2.5.21, 2.5.22, and the Hadamard–Levy theorem imply the following global inverse function theorem on the real Hilbert space.

2.5.23 Theorem. *Let $\mathbf{H}$ be a real Hilbert space and $f : \mathbf{H} \to \mathbf{H}$ be a strongly monotone C^k mapping $k \geq 1$. Then f is a C^k diffeomorphism.*

Supplement 2.5E

The Inversion Map

Let $\mathbf{E}$ and $\mathbf{F}$ be isomorphic Banach spaces and consider the inversion map $\mathfrak{I} : \mathrm{GL}(\mathbf{E}, \mathbf{F}) \to \mathrm{GL}(\mathbf{F}, \mathbf{E})$; $\mathfrak{I}(\varphi) = \varphi^{-1}$. We have shown that $\mathfrak{I}$ is C^∞ and

$$\mathbf{D}\mathfrak{I}(\varphi) \cdot \psi = -\varphi^{-1} \circ \psi \circ \varphi^{-1}$$

for $\varphi \in \mathrm{GL}(\mathbf{E}, \mathbf{F})$ and $\psi \in L(\mathbf{E}, \mathbf{F})$. We shall give below the formula for $\mathbf{D}^k\mathfrak{I}$. The proof is straightforward and done by a simple induction argument that will be left to the reader. Define the map

$$\begin{aligned}\alpha^{k+1} &: L(\mathbf{F}, \mathbf{E}) \times \cdots \times L(\mathbf{F}, \mathbf{E}) \text{ \{there are } k+1 \text{ factors\}} \\ &\to L^k(L(\mathbf{E}, \mathbf{F}); L(\mathbf{F}, \mathbf{E}))\end{aligned}$$

by

$$\begin{aligned}\alpha^{k+1}&(\chi_1, \dots, \chi_{k+1}) \cdot (\psi_1, \dots, \psi_k) \\ &= (-1)^k \chi_1 \circ \psi_1 \circ \chi_2 \circ \psi_2 \circ \cdots \circ \chi_k \circ \psi_k \circ \chi_{k+1},\end{aligned}$$

where $\chi_i \in L(\mathbf{F}, \mathbf{E})$, $i = 1, \dots, k+1$ and $\psi_j \in L(\mathbf{E}, \mathbf{F})$, $j = 1, ..., k$. Let $\mathfrak{I} \times \cdots \times \mathfrak{I}$ {with $k+1$ factors} be the mapping of $\mathrm{GL}(\mathbf{E}, \mathbf{F})$ to $\mathrm{GL}(\mathbf{F}, \mathbf{E}) \times \cdots \times \mathrm{GL}(\mathbf{F}, \mathbf{E})$ with {$k+1$ factors} defined by $(\mathfrak{I} \times \cdots \times \mathfrak{I})(\varphi) = (\varphi^{-1}, \dots, \varphi^{-1})$. Then

$$\mathbf{D}^k\mathfrak{I} = k!\, \mathrm{Sym}^k \circ \alpha^{k+1} \circ (\mathfrak{I} \times \cdots \times \mathfrak{I}),$$

where Sym^k denotes the symmetrization operator. Explicitly, for

$$\varphi \in \mathrm{GL}(\mathbf{E}, \mathbf{E}) \quad \text{and} \quad \psi_1, \dots, \psi_k \in L(\mathbf{E}, \mathbf{F}),$$

this formula becomes

$$\mathbf{D}^k\mathfrak{I}(\varphi) \cdot (\psi_1, \dots, \psi_k) = (-1)^k \sum_{\sigma \in S_k} \varphi^{-1} \circ \psi_{\sigma(1)} \circ \varphi^{-1} \cdots \circ \varphi^{-1} \circ \psi_{\sigma(k)} \circ \varphi^{-1},$$

where S_k is the group of permutations of $\{1, \dots, k\}$ (see Supplements 2.2B and 2.4A).

Exercises

◇ **2.5-1.** Let $f : \mathbb{R}^4 \to \mathbb{R}^2$ be defined by

$$f(x, y, u, v) = (u^3 + vx + y, uy + v^3 - x).$$

At what points can we solve $f(x, y, u, v) = (0, 0)$ for (u, v) in terms of (x, y)? Compute $\partial u / \partial x$.

◇ **2.5-2.** (i) Let $\mathbf{E}$ be a Banach space. Using the inverse function theorem, show that each A in a neighborhood of the identity map in $\mathrm{GL}(\mathbf{E}, \mathbf{E})$ has a unique square root.

(ii) Show that for $A \in L(\mathbf{E}, \mathbf{E})$ the series

$$\begin{aligned}B = 1 &- \frac{1}{2}(I - A) - \frac{1}{2^2 2!}(I - A)^2 - \cdots \\ &- \frac{1 \cdot 3 \cdot 5 \cdots (2n - 3)}{2^n n!}(I - A)^n - \cdots\end{aligned}$$

is absolutely convergent for $\|I - A\| < 1$. Check directly that $B^2 = A$.

◇ **2.5-3.** (i) Let $A \in L(\mathbf{E}, \mathbf{E})$ and let

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

Show this series is absolutely convergent and find an estimate for $\|e^A\|$, $A \in L(\mathbf{E}, \mathbf{E})$.

(ii) Show that if $AB = BA$, then $e^{A+B} = e^A e^B = e^B e^A$. Conclude that $(e^A)^{-1} = e^{-A}$; that is, $e^A \in \mathrm{GL}(\mathbf{E}, \mathbf{E})$.

(iii) Show that $e^{(\cdot)} : L(\mathbf{E}, \mathbf{E}) \to \mathrm{GL}(\mathbf{E}, \mathbf{E})$ is analytic.

(iv) Use the inverse function theorem to conclude that $A \mapsto e^A$ has a unique inverse around the origin. Call this inverse $A \mapsto \log A$ and note that $\log I = 0$.

(v) Show that if $\|I - A\| < 1$, the function $\log A$ is given by the absolutely convergent power series

$$\log A = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(A - I)^n.$$

(vi) If $\|I - A\| < 1$, $\|I - B\| < 1$, and $AB = BA$, conclude that $\log (AB) = \log A + \log B$. In particular, $\log A^{-1} = -\log A$.

◇ **2.5-4.** Show that the implicit function theorem implies the inverse function theorem.
Hint: Apply the implicit function theorem to $g : U \times \mathbf{F} \to \mathbf{F}$, $g(u, v) = f(u) - v$, for $f : U \subset \mathbf{E} \to \mathbf{F}$.

◇ **2.5-5.** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be C^∞ and satisfy the Cauchy–Riemann equations (see Exercise 2.3-6):

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y}, \quad \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x}.$$

Show that $\mathbf{D}f(x, y) = 0$ iff $\det(\mathbf{D}f(x, y)) = 0$. Show that the local inverse (where it exists) also satisfies the Cauchy–Riemann equations. Give a counterexample for the first statement, if f does not satisfy Cauchy–Riemann.

◇ **2.5-6.** Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = x + x^2 \cos \frac{1}{x} \quad \text{if } x \neq 0, \quad \text{and} \quad f(0) = 0.$$

Show that

(i) f is continuous;

(ii) f is differentiable at all points;

(iii) the derivative is discontinuous at $x = 0$;

(iv) $f'(0) \neq 0$;

(v) f has no inverse in any neighborhood of $x = 0$. (This shows that in the inverse function theorem the continuity hypothesis on the derivative cannot be dropped.)

◇ **2.5-7.** It is essential to have Banach spaces in the inverse function theorem rather than more general spaces such as topological vector spaces or Fréchet spaces. (The following example of the failure of Theorem 2.5.2 in Fréchet spaces is due to M. McCracken.)

Let $\mathcal{H}(\Delta)$ denote the set of all analytic functions on the open unit disk in $\mathbb{C}$, with the topology of uniform convergence on compact subsets. Let $F : \mathcal{H}(\Delta) \to \mathcal{H}(\Delta)$ be defined by

$$\sum_{n=0}^{\infty} a_n z^n \mapsto \sum_{n=0}^{\infty} a_n^2 z^n.$$

Show that F is C^∞ and that

$$\mathbf{D}F\left(\sum_{n=0}^{\infty} a_n z^n\right) \cdot \left(\sum_{n=1}^{\infty} b_n z^n\right) = \sum_{n=1}^{\infty} 2\, a_n b_n z^n.$$

(Define the Fréchet derivative in $\mathcal{H}(\Delta)$ as part of your answer.) If $a_0 = 1$ and $a_n = 1/n$, $n \neq 1$, then

$$\mathbf{D}F\left(\sum_{n=1}^{\infty} \frac{z^n}{n}\right)$$

is a bounded linear isomorphism. However, since

$$F\left(z + \frac{z^2}{2} + \cdots + \frac{z^{k-1}}{k-1} - \frac{z^k}{k} + \frac{z^{k+1}}{k+1} + \cdots\right) = F\left(\sum_{n=1}^{\infty} \frac{z^n}{n}\right)$$

conclude that F is not locally injective. (Schwartz [1967], Sternberg [1969], and Hamilton [1982] for more sophisticated versions of the inverse function theorem valid in Fréchet spaces.)

◇ **2.5-8** (Generalized Lagrange Multiplier Theorem; Luenberger [1969]). Let $f : U \subset \mathbf{E} \to \mathbf{F}$ and $g : U \subset \mathbf{E} \to \mathbf{G}$ be C^1 and suppose $\mathbf{D}g(u_0)$ is surjective. Suppose f has a local extremum (maximum or minimum) at u_0 subject to the constraint $g(u) = 0$. Then prove

(i) $\mathbf{D}f(u_0) \cdot h = 0$ for all $h \in \ker \mathbf{D}g(u_0)$, and

(ii) there is a $\lambda \in \mathbf{G}^*$ such that $\mathbf{D}f(u_0) = \lambda \mathbf{D}g(u_0)$.

(See Supplement 3.5A for the geometry behind this result).

◇ **2.5-9.** Let $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a C^1 map.

(i) Show that the set $G_r = \{\, x \in U \mid \operatorname{rank} \mathbf{D}f(x) \geq r \,\}$ is open in U.

Hint: If $x_0 \in G_r$, let $M(x_0)$ be a square block of the matrix of $\mathbf{D}f(x_0)$ in given bases of $\mathbb{R}^m$ and $\mathbb{R}^n$ of size $\geq r$ such that $\det M(x_0) \neq 0$. Using continuity of the determinant function, what can you say about $\det M(x)$ for x near x_0?

(ii) We say that R is the maximal rank of $\mathbf{D}f(x)$ on U if

$$R = \sup_{x \in U} (\operatorname{rank} \mathbf{D}f(x)).$$

Show that $V_R = \{\, x \in U \mid \operatorname{rank} \mathbf{D}f(x) = R \,\}$ is open in U. Conclude that if $\operatorname{rank} \mathbf{D}f(x_0)$ is maximal then $\operatorname{rank} \mathbf{D}f(x)$ stays maximal in a neighborhood of x_0.

(iii) Define $O_i = \operatorname{int}\{\, x \in U \mid \operatorname{rank} \mathbf{D}f(x) = i \,\}$ and let R be the maximal rank of $\mathbf{D}f(x)$, $x \in U$. Show that $O_0 \cup \cdots \cup O_R$ is dense in U.

Hint: Let $x \in U$ and let V be an arbitrary neighborhood of x. If Q denotes the maximal rank of $\mathbf{D}f(x)$ on $x \in V$, use (ii) to argue that $V \cap O_Q = \{\, x \in V \mid \operatorname{rank} \mathbf{D}f(x) = Q \,\}$ is open and nonempty in V.

(iv) Show that if a C^1 map $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ is injective (surjective onto an open set), then $m \leq n$ ($m \geq n$).

Hint: Use the rank theorem and (ii).

◇ **2.5-10** (Uniform Contraction Principle; Hale [1969], Chow and Hale [1982]).

(i) Let $T : \operatorname{cl}(U) \times V \to \mathbf{E}$ be a C^k map, where $U \subset \mathbf{E}$ and $V \subset \mathbf{F}$ are open sets. Suppose that for fixed $y \in V$, $T(x, y)$ is a contraction in x, uniformly in y. If $g(y)$ denotes the unique fixed point of $T(x, y)$, show that g is C^k.

Hint: Proceed directly as in the proof of the inverse mapping theorem.

(ii) Use (i) to prove the inverse mapping theorem.

◇ **2.5-11** (Lipschitz Inverse Function Theorem; Hirsch and Pugh [1970]).

(i) Let (X_i, d_i) be metric spaces and $f : X_1 \to X_2$. The map f is called ***Lipschitz*** if there exists a constant L such that $d_2(f(x), f(y)) \leq L d_1(x, y)$ for all $x, y \in X_1$. The smallest such L is the ***Lipschitz constant*** $L(f)$. Thus, if $X_1 = X_2$ and $L(f) < 1$, then f is a contraction. If f is not Lipschitz, set $L(f) = \infty$. Show that if $g : (X_2, d_2) \to (X_3, d_3)$, then $L(g \circ f) \leq L(g)L(f)$. Show that if X_1, X_2 are normed vector spaces and $f, g : X_1 \to X_2$, then

$$L(f + g) \leq L(f) + L(g), \quad L(f) - L(g) \leq L(f - g).$$

(ii) Let $\mathbf{E}$ be a Banach space, U an open set in $\mathbf{E}$ such that the closed ball $B_r(0) \subset U$. Let $f : U \to \mathbf{E}$ be given by $f(x) = x + \varphi(x)$, where $\varphi(0) = 0$ and φ is a contraction. Show that $f(D_r(0)) \supset D_{r(1-L(\varphi))}(0)$, that f is invertible on $f^{-1}(D_{r(1-L(\varphi))}(0))$, and that f^{-1} is Lipschitz with constant $L(f^{-1}) \leq 1/(1 - L(\varphi))$.

HINT: If $\|y\| < r(1 - L(\varphi))$, define $F : U \to \mathbf{E}$ by $F(x) = y - \varphi(x)$. Apply the contraction mapping principle in $B_r(0)$ and show that the fixed point is in $D_r(0)$. Finally, note that

$$\begin{aligned}(1 - L(\varphi))\|x_1 - x_2\| &\leq \|x_1 - x_2\| - \|\varphi(x_1) - \varphi(x_2)\| \\ &\leq \|f(x_1) - f(x_2)\|.\end{aligned}$$

(iii) Let U be an open set in the Banach space $\mathbf{E}$, V be an open set in the Banach space $\mathbf{F}$, $x_0 \in U$, $B_r(x_0) \subset U$. Let $\alpha : U \to V$ be a homeomorphism. Assume that $\alpha^{-1} : V \to U$ is Lipschitz and let $\psi : U \to \mathbf{F}$ be another Lipschitz map. Assume $L(\psi)L(\alpha^{-1}) < 1$ and define $f = \alpha + \psi : U \to \mathbf{F}$. Denote $y_0 = f(x_0)$. Show that $f(\alpha^{-1}(D_r(x_0))) \supset D_{r(1-L(\psi)L(\alpha^{-1}))}(y_0)$, that f is invertible on $f^{-1}(D_{r(1-L(\psi)L(\alpha^{-1}))}(y_0)$, and that f^{-1} is Lipschitz with constant

$$L(f^{-1}) \leq \frac{1}{L(\alpha^{-1})^{-1} - L(\psi)}.$$

HINT: Replacing ψ by the map $x \mapsto \psi(x) - \psi(x_0)$ and V by $V + \{\psi(x_0)\}$, we can assume that $\psi(x_0) = 0$ and $f(x_0) = \alpha(x_0) = y_0$. Next, replace this new f by $x \mapsto f(x + x_0) - f(x_0)$, U by $U - \{x_0\}$, and the new V by $V + \{y_0\}$; thus we can assume that

$$x_0 = 0, \quad y_0 = 0, \quad \psi(0) = 0, \quad \text{and} \quad \alpha(0) = 0.$$

Then

$$\begin{gathered} f \circ \alpha^{-1} = I + \psi \circ \alpha^{-1}, \\ (\psi \circ \alpha^{-1})(0) = 0, \\ L(\psi \circ \alpha^{-1}) \leq L(\psi)L(\alpha^{-1}) < 1, \end{gathered}$$

so (ii) is applicable.

(iv) Show that $|L(f^{-1}) - L(\alpha^{-1})| \to 0$ as $L(\psi) \to 0$. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be the homeomorphism defined by $\alpha(x) = x$ if $x \leq 0$ and $\alpha(x) = 2x$ if $x \geq 0$. Show that both α and α^{-1} are Lipschitz. Let $\psi(x) = c =$ constant. Show that $L(\psi) = 0$ and if $c \neq 0$, then $L(f^{-1} - \alpha^{-1}) \geq 1/2$. Prove, however, that if α, f are diffeomorphisms, then $L(f^{-1} - \alpha^{-1}) \to 0$ as $L(\psi) \to 0$.

◇ **2.5-12.** Use the inverse function theorem to show that simple roots of polynomials are smooth functions of their coefficients. Conclude that simple eigenvalues of operators of $\mathbb{R}^n$ are smooth functions of the operator. HINT: If $p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0$, define a smooth map $F : \mathbb{R}^{n+2} \to \mathbb{R}$ by $F(a_n, \ldots, a_0, \lambda) = p(\lambda)$ and note that if λ_0 is a simple eigenvalue, $\partial F(\lambda_0)/\partial \lambda \neq 0$.

◇ **2.5-13.** Let $\mathbf{E}, \mathbf{F}$ be Banach spaces, $f : U \to V$ a C^r bijective map, $r \geq 1$, between two open sets $U \subset \mathbf{E}$, $V \subset \mathbf{F}$. Assume that for each $x \in U$, $\mathbf{D}f(x)$ has closed split image and is one-to-one.

(i) Use the local immersion theorem to show that f is a C^r diffeomorphism.

(ii) What fails for $y = x^3$?

◇ **2.5-14.** Let $\mathbf{E}$ be a Banach space, $U \subset \mathbf{E}$ open and $f : U \to \mathbb{R}$ a C^r map, $r \geq 2$. We say that $u \in U$ is a ***critical point*** of f, if $\mathbf{D}f(u) = 0$. The critical point u is called ***strongly non-degenerate*** if $\mathbf{D}^2 f(u)$ induces a Banach space isomorphism of $\mathbf{E}$ with its dual $\mathbf{E}^*$. Use the Inverse Function Theorem on $\mathbf{D}f$ to show that strongly non-degenerate points are isolated, that is, each strongly non-degenerate point is unique in one of its neighborhoods. (A counter-example, if $\mathbf{D}^2 f$ is only injective, is given in Exercise 2.4-15.)

◇ **2.5-15.** For $u : S^1 \to \mathbb{R}$, consider the equation

$$\frac{du}{d\theta} + u^2 - \frac{1}{2\pi} \int_0^{2\pi} u^2 \, d\theta = \varepsilon \sin\theta$$

where θ is a 2π-periodic angular variable and ε is a constant. Show that if ε is sufficiently small, this equation has a solution.

◇ **2.5-16.** Use the implicit function theorem to study solvability of

$$\nabla^2 \varphi + \varphi^3 = f \;\text{ in } \Omega \quad \text{and} \quad \frac{\partial \varphi}{\partial n} = g \;\text{ on } \partial\Omega,$$

where Ω is a region in $\mathbb{R}^n$ with smooth boundary, as in Supplement 2.5C.

◇ **2.5-17.** Let $\mathbf{E}$ be a finite dimensional vector space.

(i) Show that $\det(\exp A) = e^{\operatorname{trace} A}$.

Hint: Show it for A diagonalizable and then use Exercise 2.2-12(i).

(ii) If $\mathbf{E}$ is real, show that $\exp(L(\mathbf{E}, \mathbf{F})) \cap \{ A \in \mathrm{GL}(\mathbf{E}) \mid \det A < 0 \} = \varnothing$. This shows that the exponential map is not onto.

(iii) If $\mathbf{E}$ is complex, show that the exponential map is onto. For this you will need to recall the following facts from linear algebra. Let p be the characteristic polynomial of $A \in L(\mathbf{E}, \mathbf{E})$, that is, $p(\lambda) = \det(A - \lambda I)$. Assume that p has m distinct roots $\lambda_1, \ldots, \lambda_m$ such that the multiplicity of λ_i is k_i. Then

$$\mathbf{E} = \bigoplus_{i=1}^{m} \ker(A - \lambda_i I)^{k_i} \quad \text{and} \quad \dim(\ker(A - \lambda_i I)^{k_i}) = k_i$$

Thus, to prove the exponential is onto, it suffices to prove it for operators $S \in \mathrm{GL}(\mathbf{E})$ for which the characteristic polynomial is $(\lambda - \lambda_0)^k$.

Hint: Since S is invertible, $\lambda_0 \neq 0$, so write $\lambda_0 = e^z$, $z \in \mathbb{C}$. Let $N = \lambda_0^{-1} S - I$ and

$$A = \sum_{i=1}^{k-1} \frac{(-1)^{i-1} N^i}{i}.$$

By the Cayley–Hamilton theorem (see Exercise 2.2-12(ii)), $N^k = 0$, and from the fact that $\exp(\log(1 + w)) = 1 + w$ for all $w \in \mathbb{C}$, it follows that $\exp(A + zI) = \lambda_0 \exp A = \lambda_0 (I + N) = S$.