Street, seventh floor) to give a precise location in three-dimensional space.[1] However, when Alice shows up for their meeting, Bob is nowhere to be seen. The reason, of course, is that she did not specify the *time* of their rendezvous. She should have added, "...at four o'clock." Alice needs to give *four* coordinates, three of space and one of time, to describe a precise location in a *four-dimensional* "spacetime".

Since everything that happens in the physical universe happens somewhere and somewhen, spacetime is the natural "arena" for physics and for thinking about physics. But putting space and time together into a four-dimensional spacetime is not in itself a very significant step. To see why not, consider a musical analogy.

Suppose you are playing a tune on a piano. You obviously need to know what notes to play; and of course you need to know when to play them. Thus, the natural way to describe music is in a two-dimensional space that might be called "pitchtime", containing one dimension of musical pitch and one of time. The musical staff is nothing more than a way of representing pitchtime on paper, with pitch and time on the vertical and horizontal axes, respectively.

Figure 1.1: The combination of pitch and time variables into the "pitchtime" space.

Pitchtime is a rather trivial sort of space. The two coordinates have nothing to do with one another, no *geometrical* relation. Consider two different points a and b in pitchtime two notes on the musical staff and ask, "How far apart are a and b?" A reasonable answer would be, "Notes a and b are a major fifth apart in pitch and three beats apart in time." The an-

[1] The location in question is the top floor of the Goddard Institute for Space Studies.

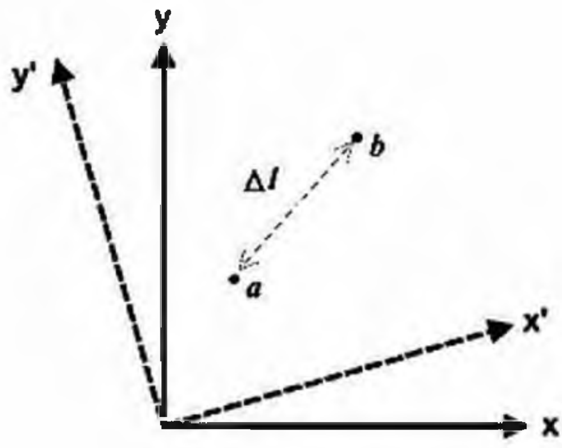

Figure 1.2: The Euclidean plane. Δl, the distance between a and b, is independent of which Cartesian coordinates are used to evaluate it.

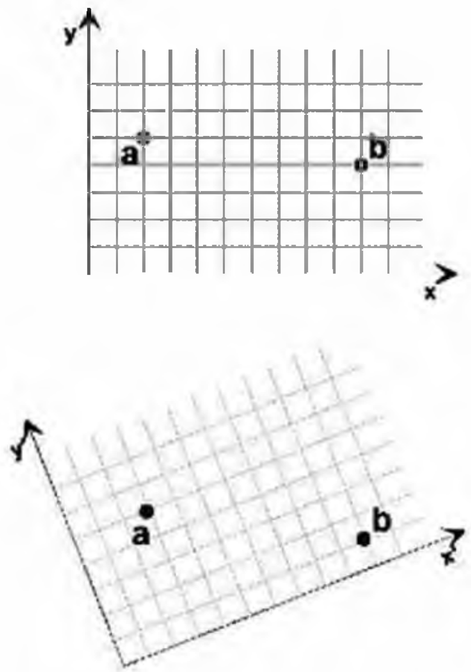

Figure 1.3: Two points in the plane, with two different coordinate systems.

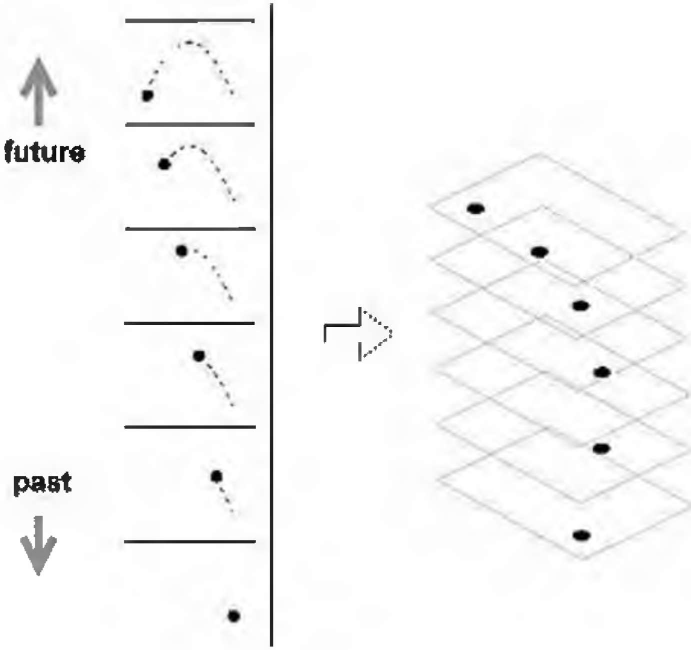

Figure 1.4: A "film" of a particle in projectile motion. The 2-D frames of the film stack up to form a 3-D spacetime diagram of the situation.

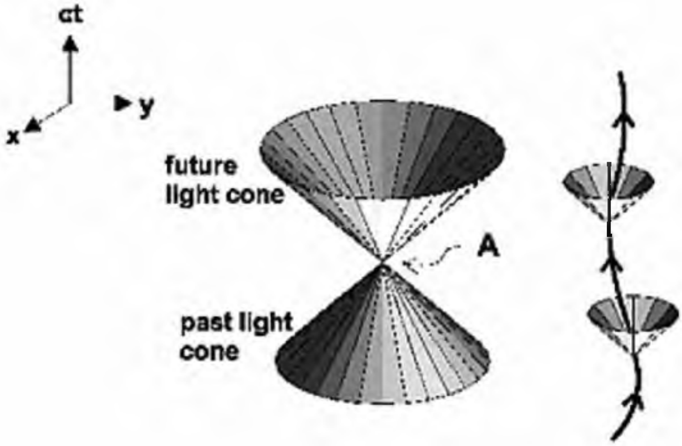

Figure 1.7: The future and past light cones of event A, and various (future) light cones of events on the world line of a particle.

expanding wavefront. In spacetime, we can say that, if the particle's world line passes through the event A, the future part of the world line must lie entirely inside the future light cone of A. Similarly, the past part of the world line must lie within the past light cone of A. In other words, the light cone marks out what parts of spacetime are "accessible" from a given point (event) in spacetime by a particle passing through that point.

A 17th-Century king of Spain, thinking about his colonies in the New World, took for granted the many weeks required to send news by ship across the sea. An event that happened yesterday in Peru, though it took place in the past, could not have influenced today's events in Spain. Similarly, though tomorrow had not yet taken place, tomorrow's events in Peru could not be influenced by today's events in Spain. Such remote but nearly contemporaneous events existed in a kind of limbo, neither in the knowable past nor in the affectable future.

The limitations of the king of Spain were merely technological. We can now exchange messages between Spain and Peru in a fraction of a second. But the limitations on travel and communication imposed by the light cone structure of spacetime are inescapable. Thus, the monarch of an interstellar empire might ponder events happening this year on a planet ten light-years away. Such events are outside of the monarch's light cone, at a spacelike with slightly different wave 4-vectors $\underline{\mathbf{k}}_1$ and $\underline{\mathbf{k}}_2$. The total wave is

$$\Phi = A\cos(k_1 z - \omega_1 t) + A\cos(k_2 z - \omega_2 t). \tag{6.19}$$

We can use trigonometric identities to rewrite this in a suggestive form. Write $\Delta k = k_2 - k_1$ and $\Delta\omega = \omega_2 - \omega_1$, and let $k = (k_1 + k_2)/2$ and $\omega = (\omega_1 + \omega_2)/2$ be the average wave vector and angular frequency. Then

$$\Phi = \underbrace{2A\cos\left(\frac{\Delta k}{2}z - \frac{\Delta\omega}{2}t\right)}_{\text{envelope}}\cos(kz - \omega t) \tag{6.20}$$

Thus, Φ is a wave characterized by ω and k, but "modulated" by the sinusoidal "envelope" function as shown.

Exercise 6.2 Derive The expression for Φ in Equation 6.20. Use the trigonometric identity

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

and write $k_1 = k - \frac{\Delta k}{2}$, $k_2 = k + \frac{\Delta k}{2}$, etc.

A snapshot of this superposition of waves is shown in Figure 6.7. It appears as a long train of pulses. The pulses themselves (as distinct from the waves within the pulses) are described by the envelope function in Equation 6.20. This function itself has the form of a wave; the wave speed of the pulses is the group velocity:

$$v_g = \frac{\Delta\omega}{\Delta k}. \tag{6.21}$$

For small differences $\Delta\omega$ and Δk (which correspond to wide pulses in the train), we can write the group velocity as a "derivative," $v_g = \frac{d\omega}{dk}$.

In general, the physics of any sort of wave will give rise to a relationship between the angular frequency ω and the wave 3-vector $\vec{\mathrm{k}}$. Usually, this can be expressed by writing ω as a function of the magnitude k of the wave 3-vector: $\omega(k)$. Such a relationship is called a *dispersion relation* for the wave. Then the phase velocity w and the group velocity v_g are

$$\begin{aligned} w &= \frac{\omega}{k} \\ v_g &= \frac{d\omega}{dk}. \end{aligned} \tag{6.22}$$