CK-12 Foundation

CK-12 Math Analysis

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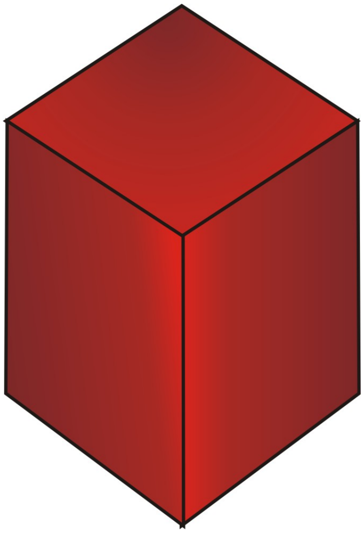
Min-Max situations

In the example in the introduction, one quantity would be maximized, while the other would be minimized. In general, however, not every situation would necessarily include both of these ideas. For example, consider a situation in which you want to make a box that has a volume of 12 cm^3, but you want the surface area of the box to be as small as possible.



In this situation, you want to find the minimum surface area, given a fixed volume.

You might also have a situation in which you are given a fixed amount of material with which to make the box, and you want to make as *large* a box as possible. In this situation, you would want to find the *maximum* volume of the box, given a fixed surface area.

Whether you will be looking for a maximum or a minimum depends on the specific situation. Consider the situations in example 1.

Example 1: In each situation determine if a quantity should be maximized or minimized.

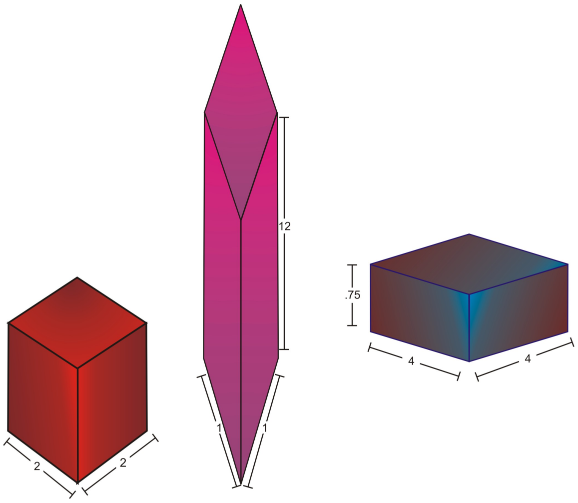
a. You have 100 feet of fence to enclose a field, and you want to create the largest field possible.
b. You run a factory that packages toilet paper, and you want to use the least amount of plastic possible for each roll.

Solution:

a. This situation involves maximizing the area of the field.
b. This situation involves minimizing the amount of plastic used per roll. (This would be the surface area of a cylinder.)

Expressions to be minimized or maximized

Consider the example above of the box with volume 12 cm^3. The volume of a box is the product of its three dimensions: length, width, and height. So we can express the volume of the box as an equation: $LWH = 12$. There are infinitely many boxes with volume 12. Three possible boxes are shown below:



We can find the surface area of each box by adding up the area of the six faces. The first box has four faces with area 6 and 2 faces with area 4, so the surface area of the box is 32 cm^2. Similarly, you can find that the second box has surface area 50cm^2. Finally the surface area of the third box is 44cm^2. Therefore, of these three boxes, the first one has the smallest surface area. However, this box does not necessarily have the smallest surface area possible for a box with the given volume. To find the box with minimum surface area requires that we know a bit more information about the box.

Let's consider a slightly simpler situation: a box with a square base and a fixed volume of 12 cm^3. Now let the length and the width of the box be x cm, and the height be h cm. We can write the volume equation as $x \cdot x \cdot h = x^2 h = 12$. We can also express the surface area in terms of x and h:

$$\text{Surface area} = S = 4xh + 2x^2$$

(Make sure you understand this formula: The base and the top are squares with area $= x^2$ and the four sides are each rectangles of area equal to xh).

We can express the surface area as a function of x if we consider the volume equation and the surface area equation as a system of equations:

$$\begin{cases} x^2 h = 12 \\ 4xh + 2x^2 = S \end{cases}$$

We want to work with the surface area equation since that is what we want to minimize. However, there are currently three variables in that equation: S, x, and h. It will be easier to graph and analyze surface area if we can express S in terms of just one other variable. So, we want to use substitution to get rid of one of the variables. We can use the volume equation to rewrite the surface area equation as a function of x.

First, rewrite the volume equation:

$x^2 h = 12 \Rightarrow h = \frac{12}{x^2}$

Now, use substitution:

$$\begin{aligned} S(x) &= 4xh + 2x^2 \\ &= 4x\left(\frac{12}{x^2}\right) + 2x^2 \\ &= \frac{48}{x} + 2x^2 \end{aligned}$$

of the approximation you use (and the number of steps you use) will depend on why you are looking for the root. For most applications coming within 0.01 of the root is a reasonable approximation, but for some applications (such as building a bridge or launching a rocket) you need much more accuracy.

Example 3

Show the first 5 iterations of finding the root of $h(x) = x^2 - x - 1$ using the starting values $a = 0$ and $b = 2$.

Solution:

1. First we verify that there is a root between $x = 0$ and $x = 2$. $h(0) = -1$ and $h(2) = 1$ so we know there is a root in the interval $[0, 2]$. Check $h\left(\frac{2+0}{2}\right) = h(1) = -1$. Since $-1 < 0$ we know the root is between $x = 1$ and $x = 2$, and we use the new interval $[1, 2]$.
2. Now we use the interval $[1, 2]$. $h\left(\frac{1+2}{2}\right) = h(1.5) = -0.25$. Since $-0.25 < 0$, we use the interval $[1.5, 2]$.
3. $h\left(\frac{1.5+2}{2}\right) = h(1.75) = 0.31$. Since $0.31 > 0$, we know that the zero is in the interval $[1.5, 1.75]$.
4. $h\left(\frac{1.5+1.75}{2}\right) = h(1.625) = 0.02$. Since $0.02 > 0$, we know the root is between 1.5 and 1.625.
5. $h\left(\frac{1.5+1.625}{2}\right) = h(1.5620) = -0.12$. Since $-0.12 < 0$, we know the root is between 1.5620 and 1.625.

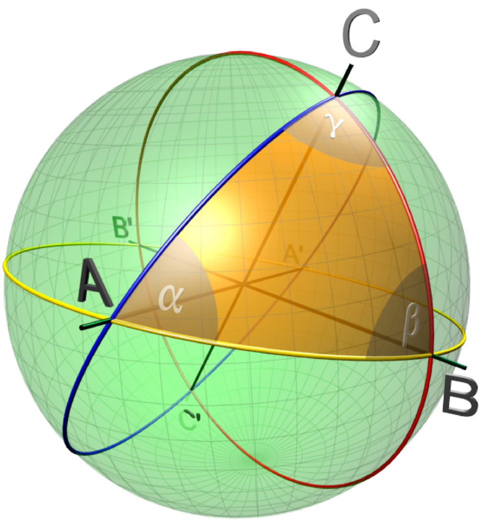
This example shows that after five iterations we have narrowed the possible location of the root to within 0.06 units. Not bad!

Recall that we have already reviewed using the **CALC** menu on a graphing calculator to find the roots of a function. This algorithm is not the one used by a calculator, but the calculator uses a similar, more efficient, algorithm for approximating the root of a function to 13 decimal places. When the calculator prompts for a **GUESS**? it is asking for a starting value to run the iterations.

Optional: An Interesting Corollary of the IVT

One surprising result of the Intermediate Value Theorem is that if you draw any great circle around the globe, then there must two antipodal points on that great circle that have exactly the same temperature.

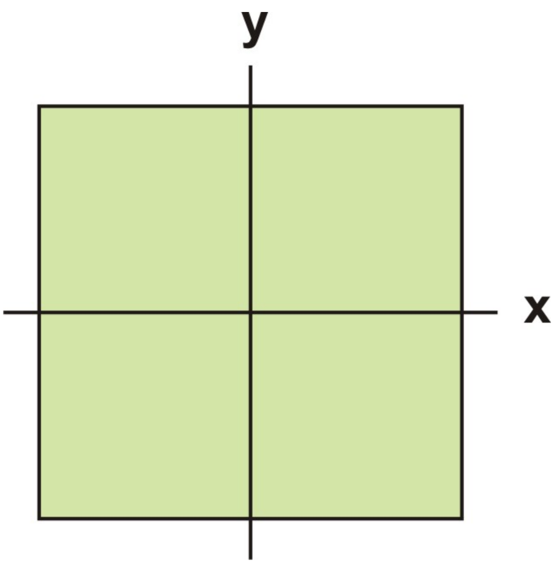
Recall that a *great circle* is a path around a sphere that gives the shortest distance between any two points on the sphere. The equator is a great circle around the globe. *Antipodal* points are two points on opposite sides of the sphere. In the diagram below, B and B' are antipodal.



(*Source:* http://commons.wikimedia.org/wiki/File:Spherical_triangle_3d_opti.png ***License:*** GNU FDL)

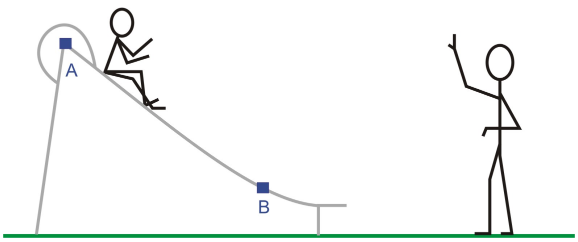
Coordinate Systems and Position Vectors in Two Dimensions

The Cartesian coordinate system used to describe a plane consists of an origin and four open axes that extend outwards from the origin towards infinity. These axes are divided into two, co-linear pairs: +**x** with −**x** and +**y** with −**y**. Usually, the x-axes are oriented horizontally or left-to-right across a page and the y-axes are oriented vertically or top-to-bottom on a printed page as shown below. The green square represents the x-y plane; remember, however, that the x-y plane is not confined to this square but rather extends outward toward infinity. On most North American maps, +**y** corresponds to North and +**x** corresponds to East. We used this coordinate system to describe Allen's trip to his mother's home across Blue Lake in one of the examples of the previous sections introducing vector addition.



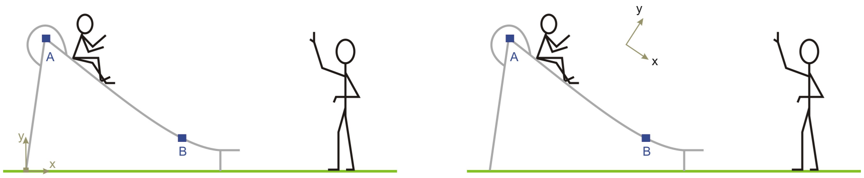
The choice of coordinate axes is arbitrary. Nothing about the motion of an object is limited by our choice of how to describe it. Therefore, we can choose a coordinate system that is most convenient. If we go back to the example of Blue Lake, we can identify the location of the Lake Park Lodge (L) with respect to Allen's home (P), to Aunt Frances's home (F), or to the Blue Lake Post Office (P). The point we choose as the reference position becomes our origin of coordinates (O). The position vector which identifies the Lodge's position within the reference frame is represented by an arrow starting at the origin and ending at the Lodge. This is like taking the rectangular coordinate system and putting the origin at a convenient location. The diagram below shows the position vector for the Lodge in three reference frames, with origins at F, A, and P respectively.

Example: The motion of an object along an inclined plane is a very common problem in introductory physics. The diagram below shows one such situation.

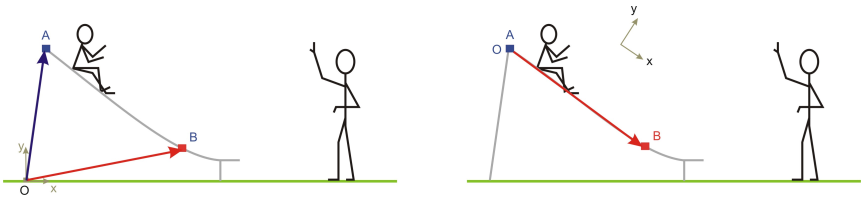


Stickman Beauford has taken his niece Brynna to the park and waves to her as she plays on the slide. Choose two coordinate systems that could be used to describe Brynna's motion and identify the position vectors for points A and B in both coordinate systems.

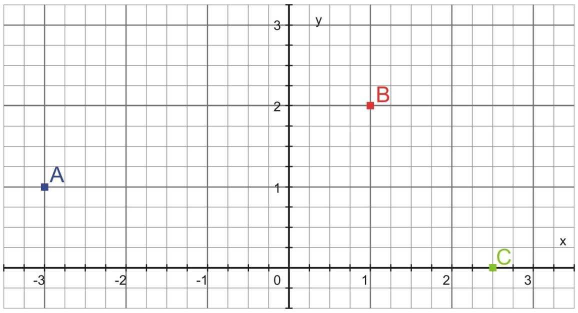
Solution: If we want to describe Brynna's motion as she moves from point A to point B along the slide, we could use a standard horizontal and vertical coordinate system with the origin at the base of the slide's ladder, but then the vector describing her motion would have components in both the x and y directions. Our mathematical description of her motion can be greatly simplified if we choose point A to be the origin and if we rotate the coordinate system such that the x-axis is parallel to the slide and the y-axis is perpendicular to the slide. Now Brynna's motion from point A to point B is only along the x-axis. Note, other choices of origin are possible.



Once we have identified an origin and coordinate axes for of our reference frames, we can use vector notation to identify the location of points A and B. The position vector for point A is the vector starting at the origin and ending at point A, $\overrightarrow{OA}$. For the standard coordinate system shown on the left above, the position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are shown on the left below. For the rotated coordinate system, shown on the right above, the position vector $\overrightarrow{OA} = 0$ and $\overrightarrow{OB} = \overrightarrow{AB}$.



Example: Identify the position vectors for the three points shown on the grid below.

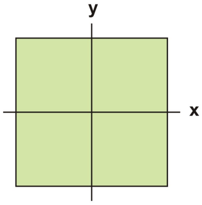


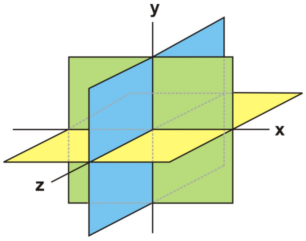
Solution: The position vectors begin at the origin, (0, 0) and end at each point:

$\overrightarrow{OA} = \langle -3, 1 \rangle$, $\overrightarrow{OB} = \langle 1, 2 \rangle$ and $\overrightarrow{OC} = \langle 2.5, 0 \rangle$

Coordinate Systems and Position Vectors in Three Dimensions

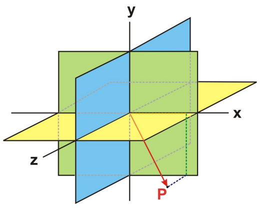
The rectangular (or Cartesian) coordinate system is used to describe a plane divided into four quadrants, as shown below left. (Note, the colored squares are used to help you visualize the space, remember that the coordinate planes actually extend outward toward infinity.)

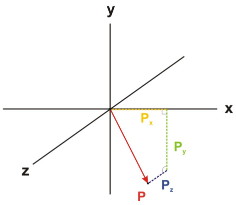




The Cartesian coordinate system use to describe three-dimensional space consists of an origin and six open axes, $+\mathbf{z}$ and $-\mathbf{z}$ are perpendicular to the **x-y** plane. These axes define three planes which divide the space into eight quadrants as shown above right. Think of these planes as cutting space three ways: left to right, top to bottom, and front to back.

By convention, we number the four quadrants of the x-y plane in this way: points in quadrant 1 have +x and +y coordinates, those in quadrant 2 have –x and +y, those in quadrant 3 have –x and –y, and those in quadrant 4 have +x and –y. There is currently no standardized numbering system for the octants in three-dimensional space, although most people identify the region with +x, +y, and +z as the first octant. The method used to identify the octants is to indicate verbally the portion of space they occupy. For example, the first octant could also be identified as (top, front, right).





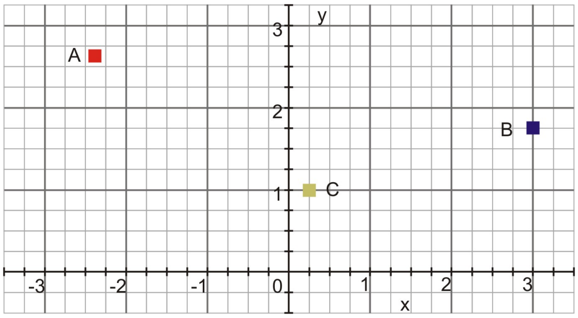
Position vectors in 3D space are still represented by arrows that begin at the origin and end at the point in question. The diagram above shows a point, P, located in the front, lower, right octant. The three components of the position vector (P_x, P_y, and P_z) are shown in the diagram. According to the Pythagorean Theorem, the magnitude of the position vector is given by:

$$|\vec{P}| = \sqrt{P_x^2 + P_y^2 + P_z^2}$$

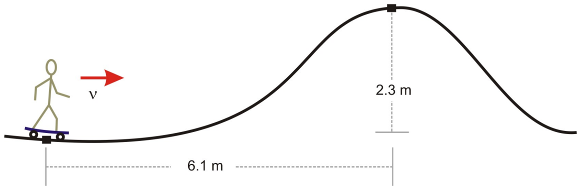
Example: Darnell was driving home from a football game in a nearby town when he swerved to avoid a deer which had run onto the road. Fortunately for Darnell, he was able to avoid hitting the deer. Unfortunately, his car ended up in the ditch beside the road. When he was unable to remove the car from the ditch by himself, he walked across a nearby field to the Tucker family farm to ask for help. The topographical map below shows Darnell's trip across the field. He traveled 300 yards south and 750 yards west from where he left his car. The map shows that he also walked uphill from an altitude of 800 feet to an altitude of 850 feet above sea level. If we treat the location of Darnell's car as the origin of coordinates, what the position vector of the Tucker farm?

Solution: Define a coordinate system where $\mathbf{x}$ = E, $\mathbf{y}$ = N, and $\mathbf{z}$ = up. Since Darnell walked south and west from the car, the $\mathbf{x}$ and $\mathbf{y}$ coordinates of the farm are both negative. If we measure all of the distances in feet (1 yard = 3 feet), the farm's position vector can be written as $\vec{P} = \left\langle P_{east}, P_{north}, P_{up} \right\rangle = \langle -2250, -900, 50 \rangle$.

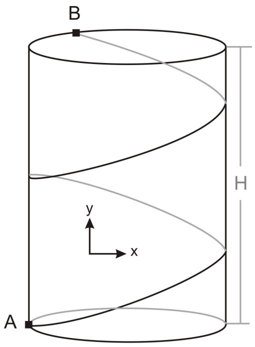
Note in this example that Darnell's walking distance was given in yards, while the elevation change was given in feet. You need to watch out for these small changes when you are solving real-life problems.



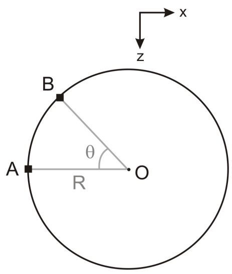
4. Determine the length of the displacement vectors from points A and C to the midpoint determined in the previous practice problem.
5. Zeke is enjoying an afternoon at the local skate-park. The diagram below shows his starting position and his ending position at the highest point on the new hill. Following the model of practice problem #1, choose two different coordinate systems which could describe this system. Find Zeke's initial and final position vectors in each of the two coordinate systems. Then identify the displacement vector from his starting position to his final position at the top of the hill.



6. An architecture student designs a spiral staircase, a model of which is shown below. The staircase winds its way around a cylinder of radius 3.5 m and height 11 m. The staircase makes 1 turns progressing counter-clockwise from its beginning at point A to its end point at B. Using an origin of coordinates at the bottom center of the staircase, determine the position vectors of points A and B. Then find the displacement vector between the two points.



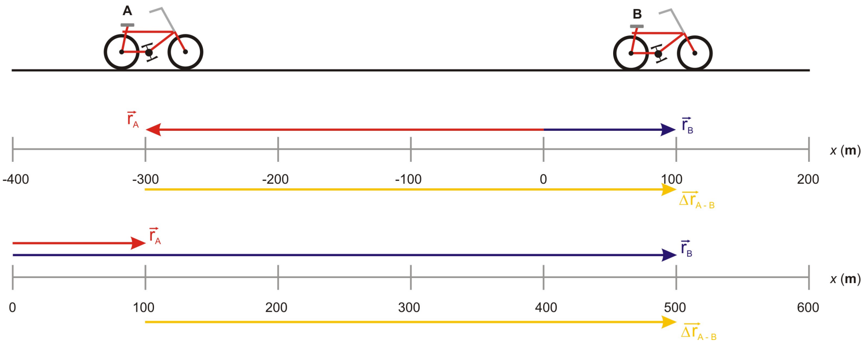
Side view

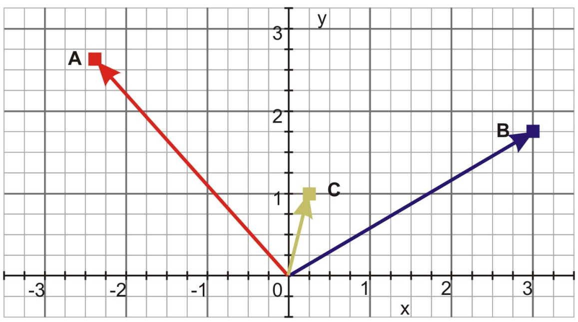


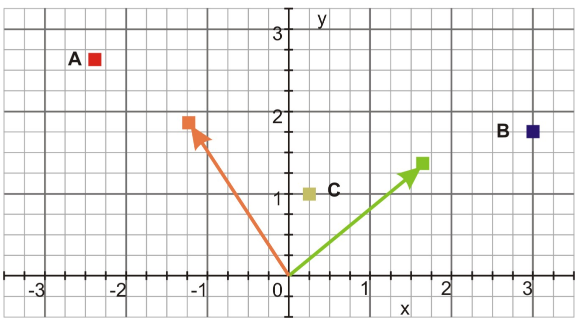
Top view

7. Identify the midpoint between points $P = (3.7, 8.4, \text{-}2.1)$ and $Q = (5.5, \text{-}1.9, \text{-}8.6)$.
8. Wilhelm and Armond head out into the marsh near their home to hunt goose for Michealmass dinner, but they disagree on the best place to set their blinds. They park their truck along the Bluffton road at an altitude of 840 m above sea level. Wilhelm heads along the riverbed, ending up 350 m west and 87 m north of the truck at an altitude of 780 m above sea level. Armond, on the other hand, heads toward another marshy area 738 m west and 92 m south of the truck at an altitude of 800 m. If the truck serves as the origin of coordinates, determine the position vector for each hunter and determine the displacement vector from Armond's position to Wilhelm's.

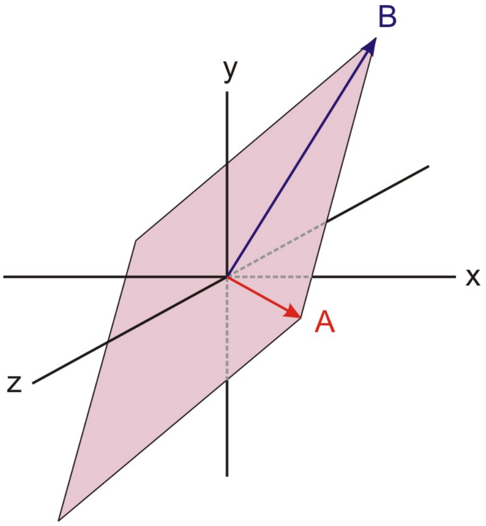
Solutions

1. The diagram shows two positions of a bicycle as it moves along a long straight road. Two possible coordinate systems for the motion are shown below. Determine the position vectors in each of the two coordinate systems for the bicycle at points A and B. Then determine the displacement vector from A to B in each case. For the upper coordinate system, the position vector of the bicycle at point A is given by $\overrightarrow{r_A} = \langle -300m, 0, 0\rangle$ and that at point B is given by $\overrightarrow{r_B} = \langle 100m, 0, 0\rangle$. This gives a displacement of $\overrightarrow{\Delta r_{A-B}} = \langle (100m - (-300m)), (0-0), (0-0)\rangle = \langle 400m, 0, 0\rangle$.
   For the upper coordinate system, the position vector of the bicycle at point A is given by $\overrightarrow{r_A} = \langle 100m, 0, 0\rangle$ and that at point B is given by $\overrightarrow{r_B} = \langle 500m, 0, 0\rangle$. This gives a displacement of $\overrightarrow{\Delta r_{A-B}} = \langle (500m - 100m), (0-0), (0-0)\rangle = \langle 400m, 0, 0\rangle$.
   

   The position vectors for the bicycle at point A are shown in red and the position vectors for point B are shown in blue. The displacement vector between points A and B is shown in gold. As you can see, the position vectors representing this motion depend on the choice of coordinate system, but the displacement vector is independent of the coordinate system. No matter how we define the origin, the bike moves 400 m in the $+x$ direction and does not move in the y or z direction.
2. Identify the position vectors for the three points shown in the diagram below.
   

   $\overrightarrow{r_A} = \langle -2.63, 2.63, 0\rangle$, $\overrightarrow{r_B} = \langle 3, 1.75, 0\rangle$, $\overrightarrow{r_C} = \langle 0.25, 1, 0\rangle$
3. Identify the midpoints between points A and C and between B and C.
   

   To find the midpoint between two points, determine the average of the two positions.
   $\overrightarrow{r_{AC,mid}} = \left\langle \frac{1}{2}(-2.63 + 0.25), \frac{1}{2}(2.63 + 1), \frac{1}{2}(0 + 0)\right\rangle = \left\langle \frac{1}{2}(-2.38), \frac{1}{2}(3.63), \frac{1}{2}(0)\right\rangle = \langle -1.19, 1.815, 0\rangle$



Example: The diagram shows two vectors **A** and **B** which define a plane passing through the origin. Use these two vectors to determine the normal vector to this plane. $\overrightarrow{A} = \langle 3, 0, 4\rangle$ and $\overrightarrow{B} = \langle 5, 10, 0\rangle$

Solution: The normal vector is defined by

$\hat{n} = \frac{\overrightarrow{U}\times\overrightarrow{V}}{|\overrightarrow{U}\times\overrightarrow{V}|}$

In this case, we obtain

$\hat{n} = \frac{\overrightarrow{A}\times\overrightarrow{B}}{|\overrightarrow{A}\times\overrightarrow{B}|}$

Use the component version of the cross-product equation to find the components of $\overrightarrow{A} \times \overrightarrow{B}$

$\overrightarrow{A} \times \overrightarrow{B} = \left\langle (A_yB_z - A_zB_y), (A_zB_x - A_xB_z), (A_xB_y - A_yB_x)\right\rangle$

$\overrightarrow{A} \times \overrightarrow{B} = \langle [(0 \cdot 0) - (4 \cdot 10)], [(4 \cdot 5) - (3 \cdot 0)], [(3 \cdot 10) - (0 \cdot 5)]\rangle$

$\overrightarrow{A} \times \overrightarrow{B} = \langle (0 - 40), (20 - 0), (30 - 0)\rangle = \langle -40, 20, 30\rangle$

Next, calculate the magnitude of the cross product, $|\overrightarrow{A} \times \overrightarrow{B}|$

$|\overrightarrow{A} \times \overrightarrow{B}| = \sqrt{(-40)^2 + 20^2 + 30^2} = \sqrt{1600 + 400 + 900} = \sqrt{2900} = 53.8516$

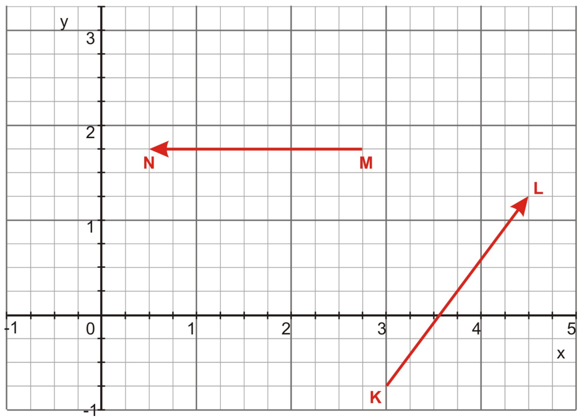
$\hat{n} = \frac{\overrightarrow{A}\times\overrightarrow{B}}{|\overrightarrow{A}\times\overrightarrow{B}|} = \frac{\langle -40,20,30\rangle}{53.9} = \left\langle \frac{-40}{53.9}, \frac{20}{53.9}, \frac{30}{53.9}\right\rangle = \langle -0.743, 0.371, 0.557\rangle$

Lesson Summary

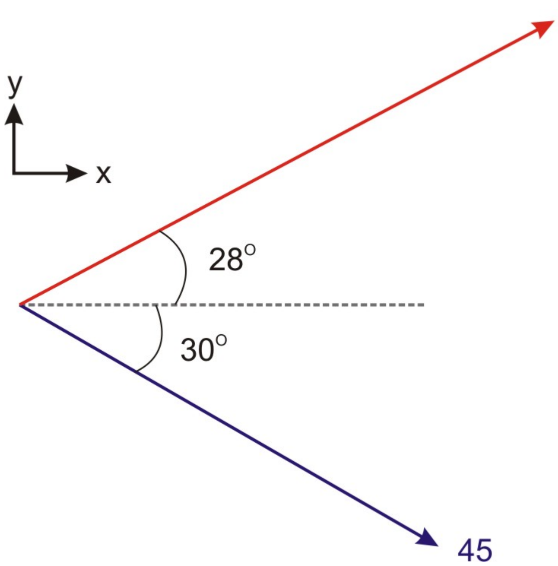
One of the two ways to multiply vector quantities is the Vector Product. The vector product, also known as the cross product, multiplies one vector by the component of the second vector which is perpendicular to the first. The result of a scalar product of two vectors is always a vector quantity which is perpendicular to the plane defined by the first two vectors. There are two ways to calculate the dot product: $\overrightarrow{A} \times \overrightarrow{B} = \left\langle (A_yB_z - A_zB_y), (A_zB_x - A_xB_z), (A_xB_y - A_yB_x)\right\rangle$ and the magnitude of the cross product is given by $|\overrightarrow{A} \times \overrightarrow{B}| = |A||B| \sin\theta$. These two versions of the cross product can be use to determine the angle between two vectors. The cross product can also be used to identify the direction perpendicular to a plane.

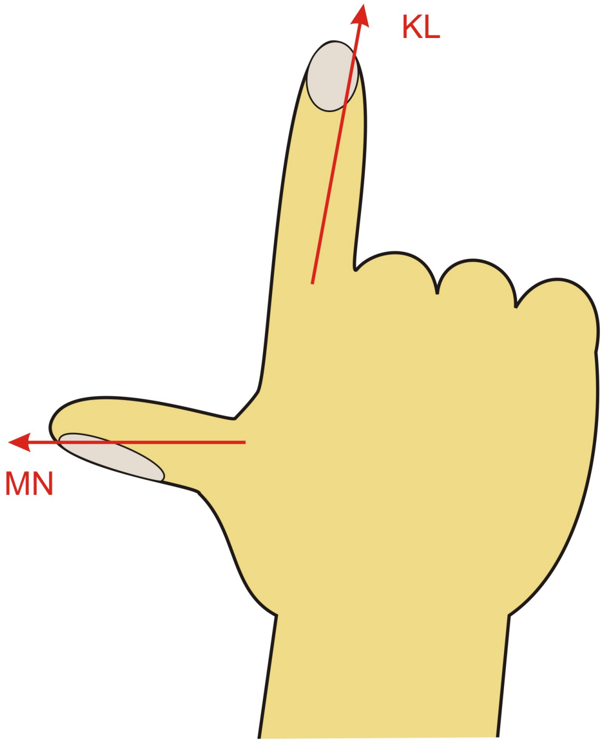
Practice Problems

1. Determine the magnitude and direction of the cross product $\vec{F} \times \vec{r}$ for the two vectors $\vec{F} = \langle 2,3,4 \rangle$ and $\vec{r} = \langle 7,6,5 \rangle$. Then use the cross product to determine the angle between the two vectors.
2. Determine the magnitude of the cross product of the two vectors shown below.



3. A plane passing through the origin is defined by the two vectors, $\vec{W} = \langle 4,5,2 \rangle$ and $\vec{L} = \langle 8,1,9 \rangle$. Determine the equation of a unit vector representing a direction perpendicular to this plane.
4. Determine the area of a parallelogram whose sides are defined by the vectors $\vec{w} = \langle 85,89,91 \rangle$ and $\vec{h} = \langle 67,70,88 \rangle$, lengths measured in centimeters.
5. Determine the magnitude of the cross product of the two vectors $\vec{f} = \langle 3,13,11 \rangle$ and $\vec{g} = \langle 9,6,15 \rangle$.
6. Determine the equation for the unit vector perpendicular to the plane defined by the two vectors $\vec{a} = \langle 2,7,4 \rangle$ and $\vec{b} = \langle 0,5,1 \rangle$.
7. Determine the area of the parallelogram whose sides are defined by $\vec{R} = \langle 27,39,52 \rangle$ and $\vec{T} = \langle 44,26,17 \rangle$, lengths measured in millimeters.
8. Determine the magnitude and direction of the cross-product of these two vectors.





3. A plane passing through the origin is defined by the two vectors, $\overrightarrow{W} = \langle 4,5,2\rangle$ and $\overrightarrow{L} = \langle 8,1,9\rangle$. Determine the equation of a unit vector representing a direction perpendicular to this plane. To solve this problem we need to use the definition of the normal vector $\hat{n} = \frac{\overrightarrow{W}\times\overrightarrow{L}}{|\overrightarrow{W}\times\overrightarrow{L}|}$, the component form of the definition of the cross product,
$\overrightarrow{W}\times\overrightarrow{L} = \left\langle (W_yL_z - W_zL_y), (W_zL_x - W_xL_z), (W_xL_y - W_yL_x)\right\rangle$. In this case, we obtain
$\overrightarrow{W}\times\overrightarrow{L} = \langle (5\cdot 9 - 2\cdot 1), (2\cdot 8 - 4\cdot 9), (4\cdot 1 - 5\cdot 8)\rangle$
$\overrightarrow{W}\times\overrightarrow{L} = \langle (45-2), (16-36), (4-40)\rangle = \langle 43, -20, -36\rangle$
We also need to know the magnitude of this cross product
$|\overrightarrow{W}\times\overrightarrow{L}| = \sqrt{x^2+y^2+z^2} = \sqrt{(43)^2+(-20)^2+(-36)^2} = \sqrt{1849+400+1296} = \sqrt{3545} = 59.54$
Now we can determine the normal vector
$\hat{n} = \frac{\overrightarrow{W}\times\overrightarrow{L}}{|\overrightarrow{W}\times\overrightarrow{L}|} = \frac{\langle 43,-20,-36\rangle}{59.54} = \left\langle \frac{43}{59.54}, \frac{-20}{59.54}, \frac{-36}{59.54}\right\rangle = \langle 0.7222, -0.3359, -0.6046\rangle$
4. Determine the area of a parallelogram whose sides are defined by the vectors $\overrightarrow{w} = \langle 85,89,91\rangle$ and $\overrightarrow{h} = \langle 67,70,88\rangle$, lengths measured in centimeters. The area of the parallelogram whose sides are defined by a pair of vectors is equal to the magnitude of the cross product of the two vectors, $|\overrightarrow{w}\times\overrightarrow{h}|$. First we need to find the cross product of the two vectors:
$\overrightarrow{w}\times\overrightarrow{h} = \left\langle (w_yh_z - w_zh_y), (w_zh_x - w_xh_z), (w_xh_y - w_yh_x)\right\rangle$
$\overrightarrow{w}\times\overrightarrow{h} = \langle (89\cdot 88 - 91\cdot 70), (91\cdot 67 - 85\cdot 88), (85\cdot 70 - 89\cdot 67)\rangle$
$\overrightarrow{w}\times\overrightarrow{h} = \langle (7832-6370), (6097-7480), (5950-5963)\rangle = \langle 1462, -1383, -13\rangle$
$|\overrightarrow{w}\times\overrightarrow{h}| = \sqrt{x^2+y^2+z^2} = \sqrt{1462^2+(-1383)^2+(13)^2} = \sqrt{4050302} \approx 2012.5$
Since the lengths of the two vectors were measured in centimeters, the area of the parallelogram is 2013 cm^2 measured to the nearest square centimeter.
5. Determine the cross product of the two vectors $\overrightarrow{f} = \langle 3,13,11\rangle$ and $\overrightarrow{g} = \langle 9,6,15\rangle$. $\overrightarrow{f}\times\overrightarrow{g} = \left\langle (f_yg_z - f_zg_y), (f_zg_x - f_xg_z), (f_xg_y - f_yg_x)\right\rangle$
$\overrightarrow{f}\times\overrightarrow{g} = \langle (13\cdot 15 - 11\cdot 6), (11\cdot 9 - 3\cdot 15), (3\cdot 6 - 13\cdot 9)\rangle$

5.5 Planes in Space

Learning objectives

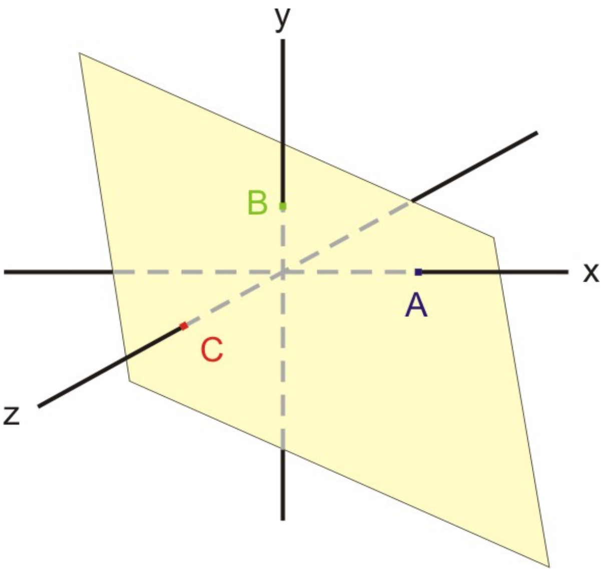
- Determine the equation of a plane given three points on the plane.
- Determine the x, y, and z intercepts of a plane.
- Determine the equation of a plane given the normal vector and a point on the plane.
- Determine the dihedral angle between two planes.

Introduction

As we have already discussed, vectors are used to indentify directions in space. We also discussed how vectors can be used to identify the orientation of a plane by identifying the direction perpendicular to that plane. In this section we will look at that calculation in reverse. Rather than determining the normal vector to a plane using two vectors which lie in that plane, we will be using the normal vector to determine the equation for the plane itself. We will also use the normal vectors to determine the intersection angle between any pair of planes.

The Equation of a Plane, Intercept Form

The diagram below shows a plane which crosses all three coordinate axes. Points A, B, and C are the locations where the plane crosses each of the coordinate axes, called *intercepts*. Their locations are given by A = (a, 0, 0), B = (0, b, 0), and C = (0, 0, c). The line segments **AB**, **BC**, and **CA** all lie in the plane. Furthermore, segment **AB** is a portion of the line of intersection between this plane and the x-y axis; segment **BC** is a portion of the line of intersection between this plane and the y-z axis; and segment **CA** is a portion of the line of intersection between this plane and the z-x axis.



The **intercept form** of the equation for a plane is given by

$1 = \frac{x}{a} + \frac{y}{b} + \frac{z}{c}$

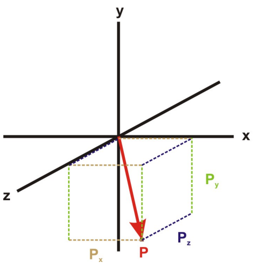
This equation must hold true for all points $(x,\ y,\ z)$ on the plane. To check this equation, insert the coordinates of point A, B, or C into the equation. For points A, B, and C respectively

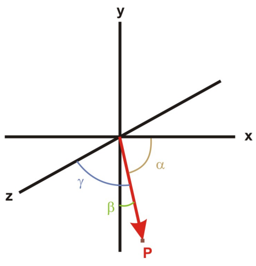
Direction Angles

As we have seen before, the equation for a vector is given by

$\overrightarrow{p} = \left\langle P_x, P_y, P_z \right\rangle$

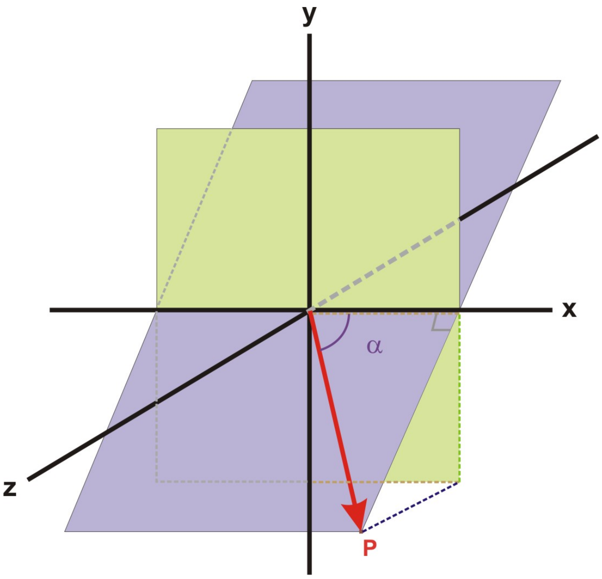
where P_{x}, P_{y}, and P_{z} are the x, y, and z coordinates of the vector obtained by projecting the vector onto the x, y, and z axes as shown below left.





The image above right, shows the angles between the position vector, $\overrightarrow{P}$, and the three axes: is the angle between $\overrightarrow{P}$ and the x-axis, is the angle between $\overrightarrow{P}$ and the y-axis, and is the angle between $\overrightarrow{P}$ and the z-axis.

The position vector, $\overrightarrow{P}$, and the unit vector, $\hat{x}$, define a plane shown in lavender below.



The direction angle, , is the angle between $\overrightarrow{P}$ and $\hat{x}$ in the plane defined by the two vectors. The other plane shown in the diagram is the X-Y plane, which was included in the diagram to help you visualize orientation of the other plane.

In our discussion of the dot product, we saw that the dot product of two vectors can be given by $\overrightarrow{A} \times \overrightarrow{B} = |\overrightarrow{A}||\overrightarrow{B}| \cos \theta$

Therefore, we can calculate the angle between $\overrightarrow{P}$ and the unit vector $\hat{x}$.

$\alpha = \cos^{-1} \frac{\overrightarrow{P} \times \hat{x}}{|\overrightarrow{P}|}$

Similarly, the direction angles and can be calculated using the equations

$\beta = \cos^{-1} \frac{\overrightarrow{P} \times \hat{y}}{|\overrightarrow{P}|}$ and $\gamma = \cos^{-1} \frac{\overrightarrow{P} \times \hat{z}}{|\overrightarrow{P}|}$